THE EQUATIONS OF PLANETARY MOTION AND THEIR SOLUTION

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ABSTRACT

Newton's original work on the theory of gravitation presented in the Principia, even in its best translation, is difficult to follow. On the other hand, in the literature of physics this theory appears only in fragments. It is because of its intellectual beauty that the author decided to compile all those fragments and present this theory in its complete version. The framework is made up of four parts: (a) setting up the differential equations that describe planetary trajectories; (b) linearising these equation; (c) providing their solution.

1. INTRODUCTION

From the world of antiquity the Greeks knew the existence of the five planets (Mercury, Venus, Mars, Jupiter, and Saturn). They generally believed that the Earth is the center of the Universe and that the sun and the planets revolve around it. But around 280 b.c., a Greek astronomer, Aristarchus of Samos (310-250 b.c.), in a revolutionary departure from the geocentric idea, he reasoned that the Earth rotates on its axis once every 24 hours, and along with the other planets it revolves around the sun once a year [1].

The above proposition was not taken seriously by the Aristarchus' contemporaries. The view that the Earth was in the center of the Universe was deeply rooted in their religious beliefs. This distorted view continued to prevail unchallenged for as many as eighteen hundred years, whereupon a Polish monk, Nicholas Copernicus (1473-1543), revived Aristarchus' proposition as a plausible explanation for the apparent movement of the planets. Copernicus was credited in the history of science as the inventor of the heliocentric idea for our solar system, but to Galileo he was just "the restorer and confirmer" [1].

Another great mind in the history of Astronomy, Johannes Kepler (1571-1630) was destined to advance Copernicus' work. Based on the experimental work of another great Danish astronomer, Tycho Brahe (1546-1601), Kepler claimed that planets prescribe elliptical rather than circular orbits around the sun, which were in perfect agreement with their apparent motion.

Aristarchus' idea, restored by Copernicus, and advanced by Kepler, culminated to its full development by the genius of Isaac Newton (1642-1727). With Kepler's empirical discoveries at hand, Newton endeavored to answer two fundamental questions. Firstly, what force causes the planets to revolve around the sun; and secondly, why their orbits are elliptical. In his pursuit to find answers to these two questions, Newton discovered the theory of *universal gravitation.*

Subsequently, based on the law of universal gravitation and his other great discovery, the second law of dynamics, Newton proved theoretically that planets do indeed prescribe elliptic trajectories around the sun.

His proof has become monumental for its ingenuity and its immense intellectual beauty. Yet, its coverage in physics textbooks is on the whole fragmented and incomplete. Resorting to the original source of this work, the arguments are found to be difficult to follow. Therefore, the author considered it worthwhile to attempt to reach Newton's conclusion via an alternate route, so that the beauty of the journey along this path be exposed to its full extent.

2. REVIEW OF KEPLER'S LAWS OF PLANETARY MOTION

Ancient Greek astronomers had observed that the apparent motion of five "stars" (in reality planets) was not following a smooth path against the background of the sky as the other stars did, but an irregular path as shown on Figure 2.1. To them, this behavior was not normal compared to the observable regular pathways of the other stars. As a result, they collectively named these five "undisciplined stars" *planets* (Greek: **πλανηται**), meaning wanderers. Individually, these five planets were named after the Greek deities (in their Latin names) Mercury, Venus, Mars, Jupiter, and Saturn. The other three planets, Uranus, Neptune, and Pluto were discovered much later after the invention of the telescope.

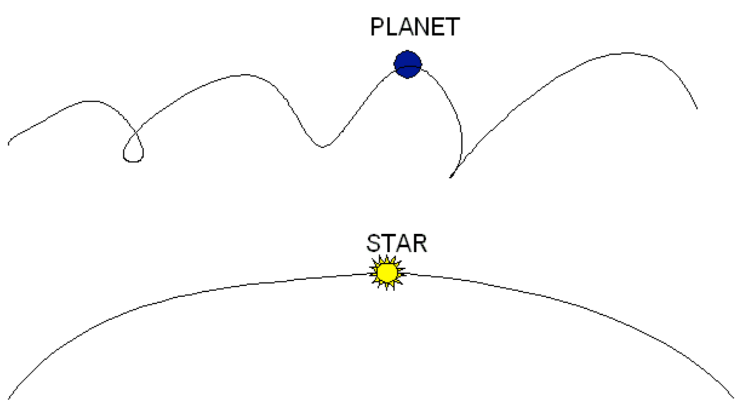

FIGURE 2.1 Apparent trajectories of a planet, and of a star

According to the Copernican heliocentric model, these planets, including the Earth, revolve around the sun in circular orbits. However, based on Tycho Brahe's observations, Kepler concluded that the speculated circular orbits were not in agreement with those observations. After twenty years of hard work, Kepler tried elliptical orbits, and to his amazement he observed an astonishing match. Kepler's conclusion from this monumental work, are consummated in his three well-known laws of planetary motion [1].

1. *The orbit of a planet is an ellipse with the sun at one of its foci.*

2. *The line joining the sun to a planet sweeps over equal areas in equal intervals of time, regardless of the length of the line.*
3. *The square of the period of any planet is proportional to the cube of its mean distance from the sun, i.e.,* $p^2 = kr^3$. *The constant k is the same for any planet.*

As it may be observed from Figure 2.2, showing the orbit of a hypothetical planet with the sun at one of the foci of the ellipse, the areas PQS and RTS generated at equal time intervals are equal.

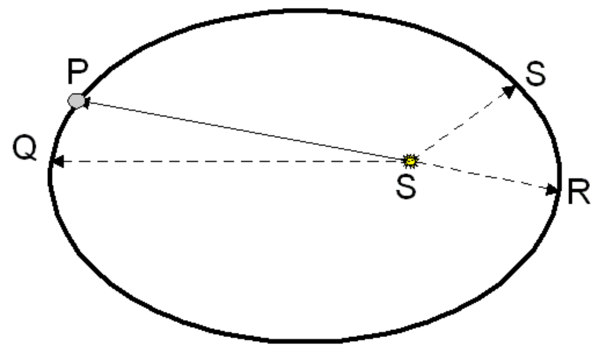

FIGURE 2.2 Typical trajectory of the orbit of a planet

3. THE LAW OF UNIVERSAL GRAVITATION

After the discovery of the laws of dynamics, Newton's subsequent great contribution to the advancement of science was his discovery of **universal gravitation.** From his law of dynamics, Newton knew that a force must act on a body in order for that body to move on a curved trajectory (e.g., along the circumference of a circle or an ellipse). After many years of intense study, and guided perhaps by Kepler's empirical laws of planetary motion, he discovered that the force responsible for the planetary trajectories was no other than the force, which is commonly known as gravity. He then concluded that this force, identified as an attraction between two masses, might be mathematically defined as follows:

Between two objects of mass *m* and *M* (Fig. 3.1), respectively, at a distance r from each other, an attractive force *F* is developed whose magnitude is proportional to each of the two masses and inversely proportional to the square of their distance "r", i.e.;

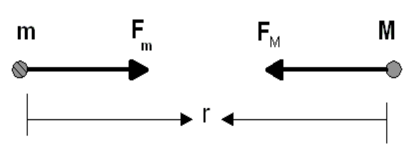

FIGURE 3.1 Gravitational attraction between two masses

$$F = G\frac{mM}{r^2} \tag{3.1}$$

where G is defined as the **constant of universal gravitation** and its value is:

$$G = 6.670 \times 10^{-11} \ \text{Newton-m}^2/\text{kg}^2$$

$$= 6.670 \times 10^{-8} \ \text{dyn-cm}_2/\text{gm}^2 \tag{3.2}$$

Equation (3.1) applies equally to a sun-planet pair and to any other pair of masses anywhere in the Universe. From Figure 3.1, and equations (3.1a) and (3.1b), it is shown that this attraction force is effected on each of the two masses. From now on, this force will be defined as *the gravitational force*.

$$\boldsymbol{F}_{\text{m}} = G\frac{mM}{r^2}\boldsymbol{u}_{\text{r}} \tag{3.1a}$$

$$\boldsymbol{F}_{M} = -G\frac{mM}{r^2}\boldsymbol{u}_{\text{r}} \tag{3.1b}$$

where $\mathbf{u}_r$ is a unit vector in the direction of *m* to *M*.

It is tempting to speculate that Newton is quite likely to have been guided by Kepler's laws in deriving the law of universal gravitation. Thus, using the circle as a special case for an ellipse, the centripetal force keeping a planet on its circular orbit is equal to the centrifugal force, that is:

$$F = \frac{m\upsilon^2}{r} \tag{3.2}$$

Taking into consideration that

$$\upsilon = \omega r = \frac{2\pi}{T}r \tag{3.3}$$

then equation (3.2) becomes:

$$F = \frac{4\pi^2 r}{T^2} \tag{3.4}$$

But in accordance with Kepler's third law, we will have:

$$T^2 = kr^3 \tag{3.5}$$

Therefore, substituting equation (3.5) into (3.4) we get:

$$F = \left(\frac{4\pi^2}{k}\right)\frac{m}{r^2} \tag{3.6}$$

Equation (3.6) is no different than equation (3.1) if we substitute $\frac{4\pi^2}{k}$ for GM where G is the gravitational constant and M is the mass of the sun.

4. GENERAL EQUATIONS OF PLANETARY MOTION IN CARTESIAN CO-ORDINATES

Shown on Figure 4.1 are two point masses m and m′ having co-ordinates in a Cartesian inertial system (i.e., orthogonal, not rotating, and not accelerating) x, y, z, and x′, y′, z′, respectively.
If the distance between m and m′ is r, then in accordance with Newton's law of gravitation the force acting on each of the masses will be:

$$F = G\frac{mm'}{r^2} \tag{4.1}$$

$$F' = -G\frac{mm'}{r^2} \tag{4.2}$$

It has been assumed that the vector **r** has a positive direction from m to m′. Therefore, as the force **F** has the direction of the vector **r** will be positive, whereas the force **F′** will be negative.
The cosines for the components F_x, F_y, and F_z of the force **F** are as follows:

$$\frac{x'-x}{r}, \quad \frac{y'-y}{r}, \quad \frac{z'-z}{r} \tag{4.3}$$

Therefore, the component forces F_x, F_y, and F_z can be defined as follows:

$$F_x = F\frac{x'-x}{r} \tag{4.4}$$

$$F_y = F\frac{y'-y}{r} \tag{4.5}$$

$$F_z = F\frac{z'-z}{r} \tag{4.6}$$

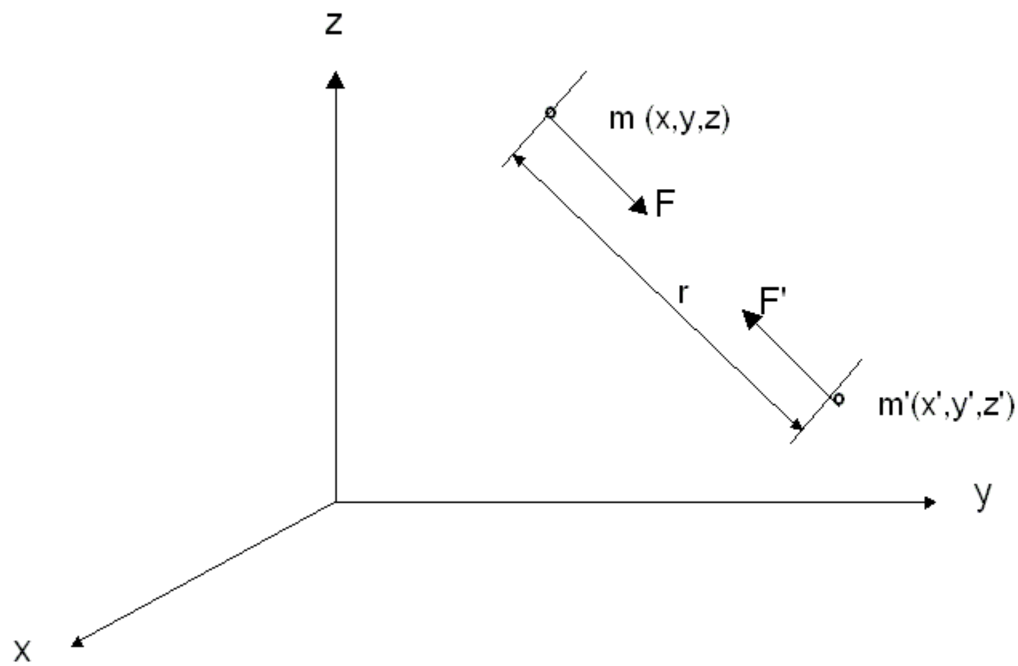

FIGURE 4.1 Illustration of two point masses under mutual gravitational attraction

Similarly, the cosines for the components of the force F′ are:

$$\frac{x-x'}{-r}, \quad \frac{y-y'}{-r}, \quad \frac{z-z'}{-r} \tag{4.7}$$

The component forces for the force F′ are:

$$F_x' = F'\frac{x-x'}{-r} = -F\frac{x-x'}{-r} = F\frac{x-x'}{r} \tag{4.8}$$

$$F_y' = F'\frac{y-y'}{-r} = -F\frac{y-y'}{-r} = F\frac{y-y'}{r} \tag{4.9}$$

$$F_z' = F'\frac{z-z'}{-r} = -F\frac{z-z'}{-r} = F\frac{z-z'}{r} \tag{4.10}$$

In accordance with Newton’s second law of dynamics, we have:

$$F_x = m\frac{d^2x}{dt^2} \tag{4.11}$$

$$F_y = m\frac{d^2y}{dt^2} \tag{4.12}$$

$$F_z = m\frac{d^2z}{dt^2} \tag{4.13}$$

Combining equations (4.11), (4.12), and (4.13) with equations (4.4), (4.5), and (4.6), we will get the equation of motion for the mass m. Thus, we will have:

$$\frac{d^2x}{dt^2} = Gm'\frac{x'-x}{r^3} \tag{4.14}$$

$$\frac{d^2y}{dt^2} = Gm'\frac{y'-y}{r^3} \tag{4.15}$$

$$\frac{d^2z}{dt^2} = Gm'\frac{z'-z}{r^3} \tag{4.16}$$

Applying the same logic for the mass m′, we will also have:

$$F_x^{'} = m'\frac{d^2x'}{dt^2} \tag{4.17}$$

$$F_y^{'} = m'\frac{d^2y'}{dt^2} \tag{4.18}$$

$$F_z^{'} = m'\frac{d^2z'}{dt^2} \tag{4.19}$$

Combining equations (4.17), (4.18), and (4.19) with (4.8), (4.9), and (4.10), we obtain the equations of motion for mass m′. Thus;

$$\frac{d^2x'}{dt^2} = Gm\frac{x-x'}{r^3} \tag{4.20}$$

$$\frac{d^2y'}{dt^2} = Gm\frac{y-y'}{r^3} \tag{4.21}$$

$$\frac{d^2z'}{dt^2} = Gm\frac{z-z'}{r^3} \tag{4.22}$$

where $$r = \sqrt{(x'-x)^2 + (y'-y)^2 + (z'-z)^2} \tag{4.23}$$

Introducing ‘j’ additional point masses, equations (4.14), (4.15), (4.16), and (4.20), (4.21), (4.22) become:

$$\frac{d^2x}{dt^2} = Gm'\frac{x'-x}{r^3} + \sum_j Gm_j\frac{x_j - x}{(r_j)^3} \tag{4.24}$$

$$\frac{d^2 y}{dt^2} = Gm' \frac{y' - y}{r^3} + \sum_j Gm_j \frac{y_j - y}{\left(r_j\right)^3} \tag{4.25}$$

$$\frac{d^2 z}{dt^2} = Gm' \frac{z' - z}{r^3} + \sum_j Gm_j \frac{z_j - z}{\left(r_j\right)^3} \tag{4.26}$$

$$\frac{d^2 x'}{dt^2} = Gm \frac{x - x'}{r^3} + \sum_j Gm_j \frac{x_j - x'}{\left(r_j'\right)} \tag{4.27}$$

$$\frac{d^2 y'}{dt^2} = Gm \frac{y - y'}{r^3} + \sum_j Gm_j \frac{y_j - y'}{\left(r_j'\right)^3} \tag{4.28}$$

$$\frac{d^2 z'}{dt^2} = Gm \frac{z - z'}{r^3} + \sum_j Gm_j \frac{z_j - z'}{\left(r_j'\right)^3} \tag{4.29}$$

where $$r_j = \sqrt{\left(x - x_j\right)^2 + \left(y - y_j\right)^2 + \left(z - z_j\right)^2} \tag{4.30}$$

and $$r_j' = \sqrt{\left(x' - x_j\right)^2 + \left(y' - y_j\right)^2 + \left(z' - z_j\right)^2} \tag{4.31}$$

To simplify the system of equations (4.24) through (4.29), we rewrite them based on a frame of reference whose the origin coincides with the mass m. The new variables X, Y, and Z are defined by the following relationships:

$$x' - x = X; \quad y' - y = Y; \quad z' - z = Z \tag{4.32a}$$

$$x_j - x = X_j; \quad y_j - y = Y_j; \quad z_j - z = Z_j \tag{4.32b}$$

By subtracting (4.32a) from (4.32b), we also get:

$$x_j - x' = X_j - X \;; \quad y_j - y' = Y_j - Y \;; \quad z_j - z' = Z_j - Z \tag{4.33}$$

Subtracting equation (4.24) from (4.27), (4.25) from (4.28), and (4.26) from (4.29); and substituting the relationships (4.32a), (4.32b), and (4.33), we get:

$$\frac{d^2X}{dt^2} = -G(m+m')\frac{X}{r^3} - \sum_j Gm_j \frac{X_j}{(r_j)^3} + \sum_j Gm_j \frac{X_j - X}{(r'_j)^3} \tag{4.34}$$

$$\frac{d^2Y}{dt^2} = -G(m+m')\frac{Y}{r^3} - \sum_j Gm_j \frac{Y_j}{(r_j)^3} + \sum_j Gm_j \frac{Y_j - Y}{(r'_j)^3} \tag{4.35}$$

$$\frac{d^2Z}{dt^2} = -G(m+m')\frac{Z}{r^3} - \sum_j Gm_j \frac{Z_j}{(r_j)^3} + \sum_j Gm_j \frac{Z_j - Z}{(r'_j)^3} \tag{4.36}$$

where

$$r = \sqrt{X^2 + Y^2 + Z^2} \tag{4.37}$$

$$r_j = \sqrt{X_j{}^2 + Y_j{}^2 + Z_j{}^2} \tag{4.38}$$

$$r_j' = \sqrt{(X - X_j)^2 + (Y - Y_j)^2 + (Z - Z_j)^2} \tag{4.39}$$

To simplify notation, we rewrite equations (4.34) through (4.39) with new variables $x = X$, $y = Y$, $z = Z$, $x_j = X_j$, $y_j = Y_j$, and $z_j = Z_j$. We also set M = m, and m′ = m. Thus we get:

Equations (4.40) through (4.45) describe the motion of a planet of mass m with respect to a sun of mass M under the interference of planets m_1, m_2,…,m_j. The variables x, y, z are the Cartesian co-ordinates of the planet m in a frame of reference whose origin is the sun. The variables x_j, y_j, and z_j are the Cartesian co-ordinates of the planet m_j in the same frame of reference. It should be noted that the equations of planetary motion were derived based on the assumption that the masses of the planets can be approximated to point masses. Because of the vast distances associated within a planetary system, this assumption is reasonable.

$$\frac{d^2x}{dt^2} = -G(M+m)\frac{x}{r^3} - \sum_j m_j \frac{x_j}{(r_j)^3} + \sum_j Gm_j \frac{x_j - x}{(r_j')^3} \qquad (4.40)$$

$$\frac{d^2y}{dt^2} = -G(M+m)\frac{y}{r^3} - \sum_j m_j \frac{y_j}{(r_j)^3} + \sum_j Gm_j \frac{y_j - y}{(r_j')^3} \qquad (4.41)$$

$$\frac{d^2z}{dt^2} = -G(M+m)\frac{z}{r^3} - \sum_j m_j \frac{z_j}{(r_j)^3} + \sum_j Gm_j \frac{z_j - z}{(r_j')^3} \qquad (4.42)$$

$$r = \sqrt{x^2 + y^2 + z^2} \qquad (4.43)$$

$$r_j = \sqrt{x_j{}^2 + y_j{}^2 + z_j{}^2} \qquad (4.44)$$

$$r_j{}' = \sqrt{(x_j - x)^2 + (y_j - y)^2 + (z_j - z)^2} \qquad (4.45)$$

5. A SIMPLIFIED EXAMPLE - A SOLAR SYSTEM WITH A SINGLE PLANET

Let us consider a solar system made of a sun and a single planet. Before we set up the equations of motion for the planet, we will prove the following law:

If a mass is moving under the influence of a central force, its trajectory will lie on the same plane.

To prove this law, we already know that the angular momentum **L** of a body A in motion under a force **F** is:

$$\frac{d\mathbf{L}}{\mathrm{dt}} = \mathbf{r} \ \mathbf{x} \ \mathbf{F} \qquad (5.1)$$

where **r** is a vector indicating the position of the body A with respect to a fixed point O at time t (see Figure (5.1).

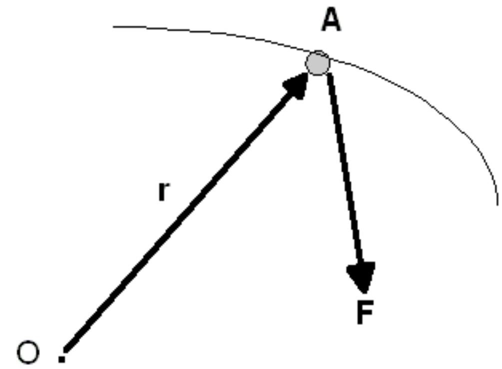

FIGURE 5.1 Illustration of a body in circular motion under the effect of a force *F*.

In the present case, the gravitational force acting on the planet is central, implying that the vector **r** and the force **F** are of the same direction. Then, in accordance with equation (5.1) above, the derivative of the angular momentum **L** will be zero. Thus:

$$\frac{d\mathbf{L}}{\mathrm{dt}} = 0 \qquad (5.2)$$

which implies that:

$$\mathbf{L} = \mathrm{m}\mathbf{r}\ \mathbf{x}\ \upsilon = \mathrm{constant} \qquad (5.2a)$$

A constant angular momentum, as shown by equation (5.2a), means that the plane of the vectors **r** and $\boldsymbol{\upsilon}$ remains the same all the time.

Now having proved that the trajectory of the planet will lie always on the same plane, we can describe it in the two-dimensional space.

Setting $m_j = 0$, and assuming that m is much smaller than M, equations (4.40) and (4.41) become:

$$\frac{d^2x}{dt^2} = -GM\,\frac{x}{r^3} \qquad (5.3)$$

$$\frac{d^2y}{dt^2} = -GM\,\frac{y}{r^3} \qquad (5.4)$$

where

$$r = \sqrt{x^2 + y^2} \qquad (5.5)$$

Equations (5.3), (5.4), and (5.5) describe the trajectory of a planet of mass m, under the effect of the gravitational force of a sun of mass M. It is assumed that the masses m and M are point masses (which is a realistic assumption, given that the distance between the two masses is very large), and that there is no interference from other planets.

6. THE EQUATIONS OF PLANETARY MOTION IN POLAR CO- ORDINATES

Equations (5.3) and (5.4) are not linear, and thus they cannot be solved analytically in their present form. Somehow, we have to convert these non-linear differential equations into linear ones by some appropriate transformation. In the pursuit of this goal, we will first convert these equations in a new form using polar co-ordinates.

The position of a particle M in the two dimensional space can be represented in polar co-ordinates by a vector of magnitude r and an angle θ, which the vector **r** forms with the x-axis (see Figure 6.1).

The velocity υ of the particle can now be analyzed into a component υ_r along the vector ***r*** and a component υ_0 perpendicular to υ_r. The relationship between Cartesian and polar co-ordinates for the position of the particle, as shown from Figure (6.1), may be described by the following equations:

$$x = r\cos\theta \qquad (6.1)$$

$$y = r\sin\theta \qquad (6.2)$$

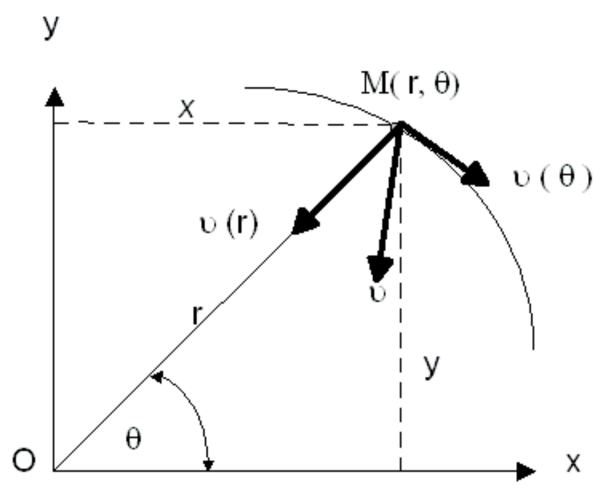

FIGURE 6.1 Analysis of velocity into polar co-ordinates

The first and second derivatives of equations (6.1) and (6.2) result in the following relationships:

$$\frac{dx}{dt} = \cos\theta\frac{dr}{dt} - r\sin\theta\frac{d\theta}{dt} \qquad (6.3$$

$$\frac{dy}{dt} = \sin\theta\frac{dr}{dt} + r\cos\theta\frac{d\theta}{dt} \qquad (6.4)$$

$$\frac{d^2x}{dt^2} = \cos\theta\frac{d^2r}{dt^2} - 2\sin\theta\frac{dr}{dt}\frac{d\theta}{dt} - r\sin\theta\frac{d^2\theta}{dt^2} - r\cos\theta\left(\frac{d\theta}{dt}\right)^2 \qquad (6.5)$$

$$\frac{d^2y}{dt^2} = \sin\theta\frac{d^2r}{dt^2} + 2\cos\theta\frac{dr}{dt}\frac{d\theta}{dt} + rco\theta\frac{d^2\theta}{dt^2} - r\sin\theta\left(\frac{d\theta}{dt}\right)^2 \qquad (6.6)$$

Substituting equations (6.5) and (6.6) into equations (6.3) and (6.4), we get:

$$\cos\theta\frac{d^2r}{dt^2} - 2\sin\theta\frac{dr}{dt}\frac{d\theta}{dt} - r\sin\theta\frac{d^2\theta}{dt^2} - r\cos\theta\left(\frac{d\theta}{dt}\right)^2 = -GM\frac{\cos\theta}{r^2} \qquad (6.7)$$

$$\sin\theta\frac{d^2r}{dt^2} + 2\cos\theta\frac{dr}{dt}\frac{d\theta}{dt} + r\cos\theta\frac{d^2\theta}{dt^2} - r\sin\theta\left(\frac{d\theta}{dt}\right)^2 = -GM\frac{\sin\theta}{r^2} \qquad (6.8)$$

To simplify equations (6.7) and (6.8), we multiply (6.7) by cosθ and (6.8) by sinθ. Then summing them up we get:

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -\frac{GM}{r^2} \qquad (6.9)$$

Now we multiply equation (6.7) by sinθ and (6.8) by cosθ, and then subtracting them we get:

$$r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt} = 0 \qquad (6.10)$$

We rewrite equations (6.9) and (6.10) as a system of two equations: Thus:

$$\boxed{\begin{array}{ll} \dfrac{d^2r}{dt^2} - r\left(\dfrac{d\theta}{dt}\right)^2 = -\dfrac{GM}{r^2} & (6.9) \\ r\dfrac{d^2\theta}{dt^2} + 2\dfrac{dr}{dt}\dfrac{d\theta}{dt} = 0 & (6.10) \end{array}}$$

Equations (6.9) and (6.10) describe the trajectory of a planet in polar co-ordinates. However, these equations are still non-linear.

Continuing our effort to transform these equations into a linear version, we will make use of our earlier conclusion that the angular momentum of a body in motion under the influence of a central force is constant (equation 5.2). Thus, we will have:

$$\boldsymbol{L} = \mathbf{m}\boldsymbol{r}\ \boldsymbol{x}\upsilon \qquad (6.11a)$$

Since

$$\boldsymbol{\upsilon} = \boldsymbol{\upsilon}_r + \boldsymbol{\upsilon}_\theta \qquad (6.11b)$$

equation (6.11a) becomes:

$$L = \mathrm{m}\boldsymbol{r}\ \boldsymbol{x}\ \boldsymbol{\upsilon}_r + \mathrm{m}\boldsymbol{r}\ \boldsymbol{x}\ \boldsymbol{\upsilon}_\theta \qquad (6.11c)$$

The term m**r x** $\boldsymbol{\upsilon}_r$ is zero because the vectors $\boldsymbol{\upsilon}_r$ and **r** have the same direction. Therefore, equation (6.11c) becomes:

$$\boldsymbol{L} = m\boldsymbol{r} \times \boldsymbol{\upsilon}_\theta \tag{6.12}$$

Since $\boldsymbol{\upsilon}_\theta$ is a vector perpendicular to **r**, the value of the angular momentum L will be:

$$L = mr\upsilon_\theta \tag{6.13}$$

Since

$$\upsilon_\theta = r\frac{d\theta}{dt} \tag{6.14}$$

then equation (6.13) becomes:

$$L = mr^2\frac{d\theta}{dt} \tag{6.15}$$

Solving equation (6.15) for dθ/dt, we get:

$$\frac{d\theta}{dt} = \frac{L}{mr^2} \tag{6.16}$$

where the value L of the angular momentum is constant.

Substituting equation (6.16) into equation (6.9), we get:

$$\frac{d^2r}{dt^2} - \frac{L^2}{m^2r^3} = -\frac{GM}{r^2} \tag{6.17}$$

Equation (6.17) contains only one variable (r) but it is still non-linear as it contains powers of r.
We also make the observation that equation (6.10) is the derivative of equation (6.16). Thus, taking the derivative of equation (6.16), we get:

$$\frac{d^2\theta}{dt^2} = -2\frac{L}{m}\frac{1}{r^3}\frac{dr}{dt}$$

or

$$r\frac{d^2\theta}{dt^2} + 2\left(\frac{L}{mr^2}\right)\frac{dr}{dt} = 0$$

or

$$r\frac{d^2\theta}{dt^2} + 2\frac{d\theta}{dt}\frac{dr}{dt} = 0 \tag{6.10}$$

Therefore, the system of equations (6.9) and (6.10) is now simplified to:

$$\boxed{\begin{array}{ll} \dfrac{d^2r}{dt^2} - \dfrac{L^2}{m^2}\dfrac{1}{r^3} = -\dfrac{GM}{r^2} & (6.17) \\[2ex] \dfrac{d\theta}{dt} = \dfrac{L}{m}\dfrac{1}{r^2} & (6.16) \end{array}}$$

In our continuing effort to transform these equations into a linear form, we will make the following transformation [2].

$$r = \frac{1}{u} \qquad (6.18)$$

To combine the two equations into one, we will eliminate time, and thus we will be able to express r as a function of θ. Thus, by differentiating (6.18), we get:

$$\frac{dr}{dt} = -\frac{1}{u^2}\frac{du}{d\theta}\frac{d\theta}{dt} \qquad (6.19)$$

Combining (6.16) with (6.19), we get:

$$\frac{dr}{dt} = -\frac{1}{u^2}\frac{du}{d\theta}\frac{L}{mr^2} \qquad (6.20)$$

Combining (6.20) with (6.18), we get:

$$\frac{dr}{dt} = -\frac{L}{m}\frac{1}{u^2}\frac{du}{d\theta}u^2 \qquad (6.21)$$

or

$$\frac{dr}{dt} = -\frac{L}{m}\frac{du}{d\theta} \qquad (6.21a)$$

Differentiating also (6.22), we get:

$$\frac{d^2r}{dt^2} = -\frac{L}{m}\frac{d^2u}{d.\theta^2}\frac{d\theta}{dt} \qquad (6.23)$$

Combining (6.23) with (6.16) and (6.18), we get:

$$\frac{d^2r}{dt^2} = -\left(\frac{L}{m}\right)^2 u^2 \frac{d^2u}{d\theta^2} \qquad (6.23a)$$

Substituting equations (6.23a) and (6.18) into (6.17), we get:

$$\boxed{\frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2}} \tag{6.24}$$

where
$$u = \frac{1}{r} \tag{6.18}$$
$$h = \frac{L}{m} \tag{6.25}$$
$$\mu = GM \tag{6.26}$$

Equation (6.24) is a linear second order differential equation, which can now be solved analytically.
It is remarkable what has been achieved with a simple transformation (Eq. 6.18). A highly non-linear differential equation is transformed into an ordinary linear one! It is tempting here to speculate whether this was simply by accident or perhaps that natural phenomena are destined to be represented only by linear differential equations.

7. SOLUTION OF THE EQUATION OF PLANETARY MOTION FOR A SOLAR SYSTEM OF A SINGLE PLANET (EQUATION 6.24)

The general solution of equation (6.24) will be the sum of the complementary solution u_c and the particular integral u_p. The complementary solution u_c is the solution of the homogeneous equation, that is:

$$\frac{d^2u}{d\theta^2} + u = 0 \tag{7.1}$$

Defining the operator D = d/dθ, equation (7.1) may be rewritten as follows:

$$\left(D^2 + 1\right)u = 0 \tag{7.2}$$

or
$$(D + i)(D - i)u = 0 \tag{7.3}$$

The general solution of equation (7.3) is the following:

$$u_c = c_1 e^{-i\theta} + c_2 e^{i\theta} \tag{7.4}$$

The particular integral u_p may be computed as follows:

We rewrite equation (7.24) using the operator D. Thus;

$$\left(D+i\right)\left(D-i\right)u_p = \frac{\mu}{h^2} \tag{7.5}$$

Solving for (D+i)u, we get:

$$\left(D+i\right)u_p = \frac{1}{D-i}f\left(x\right) \tag{7.6}$$

where
$$f\left(x\right) = \frac{\mu}{h^2} \tag{7.7}$$

From standard textbooks on differential equations, integration of the right-hand side of equation (7.6) using the operator 1/(D- i) results in the following:

$$\left(D+i\right)u_p = -\frac{\mu e^{i\theta}}{ih^2}\int e^{-i\theta}d\left(-i\theta\right) \tag{7.8}$$

or
$$\left(D+i\right)u_p = i\frac{\mu}{h^2}e^{i\theta}e^{-i\theta}$$

or
$$\left(D+i\right)u_p = i\frac{\mu}{h^2} \tag{7.9}$$

Solving for u_p, the particular integral may be computed as follows:

$$u_p = \frac{1}{D+i}\frac{\mu}{h^2}i$$

or
$$u_p = e^{-i\theta}\int\left(i\frac{\mu}{h^2}\right)e^{i\theta}d\theta$$

or
$$u_p = \frac{\mu}{h^2}e^{-i\theta}\int e^{i\theta}d\left(i\theta\right)$$

or
$$u_p = \frac{\mu}{h^2}e^{-i\theta}e^{i\theta}$$

or
$$u_p = \frac{\mu}{h^2} \tag{7.10}$$

Therefore, the general solution of equation (7.24) is:

$$u = u_c + u_p = c_1 e^{i\theta} + c_2 e^{-i\theta} + \frac{\mu}{h^2} \tag{7.11}$$

Using the relationships:

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{7.12}$$

and

$$e^{-i\theta} = \cos\theta - i\sin\theta \tag{7.13}$$

equation (7.11) becomes:

$$u = (c_1 + c_2)\cos\theta + i(c_1 - c_2)\sin\theta + \frac{\mu}{h^2} \tag{7.14}$$

or

$$\frac{1}{r} = (c_1 + c_2)\cos\theta + (c_1 - c_2)\sin\theta + \frac{\mu}{h^2} \tag{7.14a}$$

Since the left-hand side is a real number, the right-hand side must be a real number as well. Therefore, the imaginary term must be zero, implying that c_1-c_2=0, or c_1 = c_2= C/2. Then, equation (7.14a) becomes:

$$\boxed{\frac{1}{r} = C\cos\theta + \frac{\mu}{h^2}} \tag{7.15}$$

Equation (7.15) constitutes the general solution of the equation of planetary motion (6.24).

7.1 determination of the constant ‘C’ in equation (7.15)

Let the velocity and the position vector at time t=0 be υ_o and r_0, respectively. From equation (7.15) we get:

$$\cos^2\theta = \frac{\left(\frac{1}{r_o} - \frac{\mu}{h^2}\right)^2}{C^2} \tag{7.1.1}$$

Also, by differentiating equation (7.15) we get:

$$-\frac{1}{r^2}\left(\frac{dr}{dt}\right) = -C\sin\theta\frac{d\theta}{dt}$$

or

$$\left(\frac{dr}{dt}\right) = Cr^2\sin\theta\frac{d\theta}{dt} \tag{7.1.2}$$

Taking into consideration equation (6.16), equation (7.1.2) becomes:

$$\left(\frac{dr}{dt}\right) = Ch\sin\theta \tag{7.1.3}$$

from which we will have:

$$(\sin\theta)^2 = \frac{\left(\frac{dr}{dt}\right)^2}{C^2h^2} \tag{7.1.4}$$

Summing up equations (7.1.1) and (7.1.4) we will have:

$$\frac{\left(\frac{1}{r_o}\right)^2 + \left(\frac{\mu}{h^2}\right)^2 - \frac{2\mu}{r_o h^2}}{C^2} + \frac{\left(\frac{dr}{dt}\right)^2}{C^2h^2} = 1 \tag{7.1.5}$$

Now we will compute $(dr/dt)^2$ by using the principle of conservation of energy. Thus, we will have:

$$\begin{aligned} E &= \frac{1}{2}m\upsilon_o{}^2 - \frac{m\mu}{r_o} \\ &= \frac{1}{2}m\left(\upsilon_{or}{}^2 + \upsilon_{o\theta}{}^2\right) - \frac{m\mu}{r_o} \\ &= \frac{1}{2}m\left[\left(\frac{dr}{dt}\right)_o^2 + r_o{}^2\left(\frac{d\theta}{dt}\right)_o^2\right] - \frac{m\mu}{r_o} \\ &= \frac{1}{2}m\left[\left(\frac{dr}{dt}\right)_o^2 + r_o{}^2\frac{L^2}{m^2r_o{}^4}\right] - \frac{m\mu}{r_o} \end{aligned} \tag{7.1.6}$$

Solving equation (7.1.6) for $(dr/dt)^2$, we get:

$$\left(\frac{dr}{dt}\right)_o^2 = \frac{2E}{m} + \frac{2\mu}{r_o} - \frac{h^2}{r_o{}^2} \tag{7.1.7}$$

Substituting (7.1.7) into (7.1.5), we get:

$$\frac{\frac{1}{r_o{}^2}+\frac{\mu^2}{h^4}-\frac{2\mu}{r_o h^2}}{C^2}+\frac{\frac{2E}{m}+\frac{2\mu}{r_o}-\frac{h^2}{r_o{}^2}}{C^2 h^2}=1 \tag{7.1.8}$$

or

$$\frac{\mu^2}{h^2}+\frac{2E}{m}=C^2h^2 \tag{7.1.9}$$

Solving equation (7.1.9) for C, we get:

$$C=\frac{\mu}{h^2}\sqrt{1+\frac{2Eh^2}{m\mu^2}} \tag{7.1.10}$$

Substituting equation (7.1.10) into equation (7.15), we get the equation of planetary motion, that is:

$$\boxed{\frac{1}{r}=\frac{\mu}{h^2}\sqrt{1+\frac{2Eh^2}{m\mu^2}}\cos\theta+\frac{\mu}{h^2}} \tag{7.1.11}$$

Equation (7.1.11) can be written in a more simplified form as follows:

$$\frac{1}{r}=\frac{1}{ef}\left(1+e\cos\theta\right) \tag{7.1.12}$$

Equation (7.1.12) is the familiar expression of a conic section, where:

$$e=\sqrt{1+2\frac{Eh^2}{m\mu^2}} \tag{7.1.13}$$

$$f=\frac{h^2}{\mu\sqrt{1+2\frac{Eh^2}{m\mu^2}}} \tag{7.1.14}$$

$$h=\frac{L}{m} \tag{6.25}$$

$$\mu=GM \tag{6.26}$$

and

$$E=\frac{1}{2}m\upsilon^2-\frac{mGM}{r} \tag{7.1.15}$$

If the value of e is less than 1, the conic section is an ellipse, if it is equal to 1 is a parabola, and if it is greater than 1 is a hyperbola. Obviously, from equation (7.1.13) it is inferred that the total energy E must be negative and the expression under the radical positive in order for e to be positive and less than 1. Indeed, based on the equations (7.1.13) through (7.1.15), and (6.25) and (6.26), using the values: r=1.49x10^{11} m, m =5.98x10^{24} kg, M=1.98x10^{30} kg, T=3.16x10^{7} sec, and G=6.67x10^{-11} Newton-m^2/kg^2, it is easily proven that E is negative and with absolute value less than one, resulting a positive and less than one e. Therefore, equation (7.1.11) describes an ellipse.

8. VERIFICATION OF KEPLER'S LAWS

Kepler's first law, that planets prescribe elliptical orbits around the sun, is self–evident from equation (7.1.11), which has been shown above to be indeed an ellipse.

Kepler's second law can be verified as follows: The area that the radius r sweeps over between times t and $t+\Delta t$ is:

$$dA = \frac{1}{2}(rd\theta)r = \frac{1}{2}r^2 d\theta$$

or

$$dA = \frac{1}{2}r^2 \frac{d\theta}{dt}dt \tag{8.1}$$

Combining equation (8.1) with equation (7.1.12) and integrating, we get:

$$A_{12} = \int_{t_1}^{t_2}\left(\frac{e^2 f^2}{(1+e\cos\theta)^2}\frac{d\theta}{dt}\right)dt \tag{8.2}$$

To compute this integral is not an easy task. Therefore, we leave this challenge to the mathematicians. However, there is an easier way to prove Kepler's second law.

Integrating equation (8.1) between time t_1 and time t_2, we get:

$$A_{12} = \int_{t_1}^{t_2}\left(r^2 \frac{d\theta}{dt}\right)dt \tag{8.3}$$

To verify this law requires that the quantity resulting from the above integration be a function of the time difference (t_2 - t_1) only. But for this to happen, the integrand in equation (8.3) must have a constant value. Indeed, in accordance with equation (6.15), we have:

$$r^2 \frac{d\theta}{dt} = \frac{L}{m} \tag{8.4}$$

Therefore, the integral in equation (8.3) becomes:

$$A_{12} = \frac{L}{m}(t_2 - t_1) \tag{8.5}$$

Equation (8.5) states that the area A_{12} swept over by the radius r depends on the time interval (t_2-t_1) only, regardless at which point of the ellipse the starting time is taken, which verifies Kepler's second law. It is reminded that the angular momentum L is constant (see section 5.0).

With regard to Kepler's third law, given that the radius of an ellipse is not constant, it cannot be verified analytically, except in the special case where the eccentricity is very small and the ellipse can be approximated to a circle.

9. BIBLIOGRAPHY

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