A guide to computing orbital positions of major solar system bodies: forward and inverse calculations.

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The well known inverse-square law of gravitational attraction between masses, and Newton's laws of classical mechanics, together provide a basis to calculate positional aspects of any body in the solar system. Although the concepts of such a calculation are, in principle, simple, the three dimensional nature of the problem often leads to difficulties in the application of the associated geometry. In a one (or two) body orbital problem, e.g. that of a planet and the Sun, the problem reduces to Kepler's equation. In these notes, Kepler's equation is reviewed and how its 'elliptical-orbit' solution, typically a good approximation for most planets in our solar system, is described by six orbital parameters. For purposes of observing major bodies of the solar system, including the planets, we show practically how it is possible to calculate time dependent positions for such bodies from an Earth observer's perspective. The final part of this solution involves a transformation from celestial ("right ascension" and "declination") co-ordinates to local ("altitude" and "azimuth") co-ordinates. This is useful in its own right also for the observation of stars, nebulae and galaxies. The inverse to the Kepler problem is the determination of orbital parameters from a set of planetary observations. The method of least squares is used for this purpose and is presented in the final section.

Keywords: Astronomy, Kepler's equation, Orbital parameters, Co-ordinate transformation, Inverse Kepler problem.

Contents

1. The Elliptical orbit in a plane

Gravity is a central force. Consequently, the gravitational orbit of one body about another is constrained to planar geometry[1]. This provides a major simplification to the analysis of orbital motion since the problem can be (for now) expressed in two dimensions. Consider a system of two bodies, of masses M_s and m_p (e.g. the Sun and a planet respectively). Assuming that the orbital motion of each is bound, then the orbits of each about their common centre of mass form ellipses. We will assume that $M_s >> m_p$, so that the centre of mass can be assumed to lie at the position of mass M_s (representative of the Sun) and mass m_p (a planet) would trace the elliptical orbit (fig. 1). Any elliptical orbit is specified by various attributes which are now defined. Let the centre of the ellipse be at point C. F_1 and F_2 are special points arranged symmetrically about C and describe the foci of the ellipse. Mass M_s is positioned, not at C, but at one of the foci. We choose F_2 to be the *occupied* focus. P on the ellipse is the point of closest approach to F_1 and is called the *perihelion*. Let Q and R be two points on the ellipse. Q is arbitrary and R is positioned such that line CR is perpendicular to line CP. The length CR, called the *semi-minor* axis, is of length b and CP, the *semi-major* axis, has length a. These lengths are related via the eccentricity of the ellipse, e,

$$b = a\sqrt{1 - e^2}, \tag{1}$$

which, in the special case of a circular orbit is zero ($b = a$). The distance between the centre of the orbit and a focus is $CF_2 = ea$.

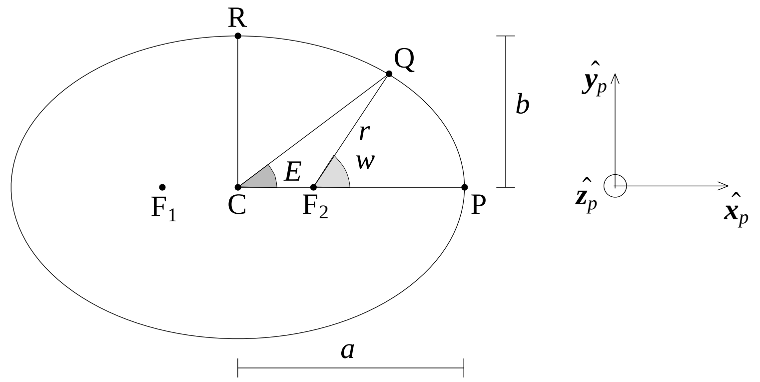


Fig. 1: An ellipse described by its centre, C, two foci F_1 and F_2, the latter being occupied, and semi-axes of lengths a and b. P is the perihelion, Q is an arbitrary point on the ellipse (a distance r from F_2) and R is the point where the semi-minor axis meets the ellipse. E and w are two possible angles describing Q. The co-ordinate system is defined by the three unit vectors $\hat{\boldsymbol{x}}_p$, $\hat{\boldsymbol{y}}_p$ and $\hat{\boldsymbol{z}}_p$ (p = planetary co-ordinates).

In order to determine the position of mass m_p on the ellipse at a given time t, one should solve Kepler's equation [1], expressed as,

$$M = E - e \sin E. \tag{2}$$

[1] For the many (more than two) -body problem, the planar constraint does not generally hold. In the configuration of our solar system, where interaction between each body and the Sun dominates, the system can be approximated as a collection of independent pairs of bodies.

In eq. (2) there are two new parameters, M and E. M, called the *mean anomaly*, is defined as,

$$M = n(t - T), \tag{3}$$

where n is the *mean motion*, $n = 2\pi/P$ (P is the period of the orbit) and $t - T$ is the time elapsed since perihelion. The angle E is the *eccentric anomaly*. This is illustrated in fig. 1 for point Q on the ellipse (angle PCQ). Eq. (2) is non-linear in the unknown E and cannot be solved analytically. It is possible however to gain an approximate solution. One way of achieving this to arbitrary precision is by numerical means with the Newton-Raphson iterative method [2].

Once the eccentric anomaly has been determined, it is necessary to compute another angle associated with E. The *true anomaly*, w, is the angle between the perihelion and the point Q about the occupied focus. This angle is the one which interests us and is related to E via the trigonometric identity [1],

$$\tan\frac{w}{2} = \left(\frac{1 + e}{1 - e}\right)^{1/2} \tan\frac{E}{2}, \tag{4}$$

and the distance, r, of Q from F_2 is,

$$r = a(1 - e\cos E). \tag{5}$$

Together, the time dependent variables w and r allow the position vector of the body to be determined with respect to the occupied focus. Declaring, $\boldsymbol{r}_p$ as the position vector in the planetary co-ordinate system defined in fig. 1, it is expressed,

$$\boldsymbol{r}_p = \begin{pmatrix} r\cos w \\ r\sin w \\ 0 \end{pmatrix}. \tag{6}$$

We introduce the notation that a vector expressed in plane parentheses (as in eq. (6)) shall be expressed in the planetary co-ordinate system. Three of the six parameters which fully describe the orbit (a, e and T) have now been introduced. Although other parameters arise in the above, they are not independent and may be inferred from relations (e.g. b may be found from eq. (1) and P can be calculated from $P = 2\pi\sqrt{a^3/GM_s}$ [1]). The remaining three parameters are described in section 2 and specify how the ellipse is orientated in space. Numerical values of six quantities for most planets of our solar system are listed in section 7.

2. The transformation to ecliptic co-ordinates

Observations of the planets are made from Earth and so we must move to a co-ordinate system which is convenient to an Earth-bound perspective (*geocentric co-ordinates*). This is the combined aim of sections 2 and 3 of these notes. The information known from the last section consists of the position vector of the body at a given time expressed in the planetary co-ordinate system (fig. 1). The planetary co-ordinate system is specific to a particular planet and so we should transfer to a representation which is common to all objects. There are many stages in the transformation. The first one which we shall do is to convert to *ecliptic co-ordinates*. The ecliptic is the plane of the Earth's orbit and contains the x-y plane of the ecliptic co-ordinate system. With the centre of the system on the Sun, the x-axis points in the direction of the *vernal equinox*, and the z-axis points perpendicular to the orbital plane (looking along z, the Earth's orbit is clockwise) and the y-axis is perpendicular to the other two axes in a right-handed sense (fig. 2).

Three angles denote the orientation of the orbit with respect to the ecliptic, Ω, i and ω (the *longitude of the ascending node*, the *inclination* and the *argument of the perihelion* respectively). In addition to the three orbital parameters defined in section 1, these angles complete the orbital definition. All six parameters have now been mentioned. When finding numerical values for the planets, note that some references list an alternative set of parameters. For example, instead of T, the time of perihelion (in eq. (3)), it is usual to quote the *mean longitude*, ε (or more formally called the *mean longitude at the epoch*). To find the mean anomaly, the following formula,

$$M = n(t - t_0) - \varpi + \varepsilon, \tag{7}$$

should be used, where $\varpi = \omega + \Omega$. ϖ is called the *longitude of the perihelion*. In eq. (7), t_0 specifies the moment in time (epoch) associated with the given value of ε. It is the alternative orbital parameter ε which is listed with the other parameters in section 7 for most planets of the solar system.

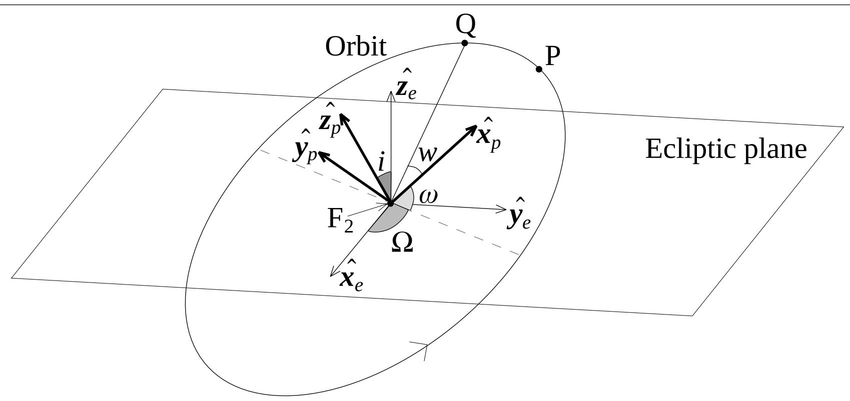


Fig. 2: The orbit of a planet (ellipse) relative to the Earth (the Earth's orbit is in the ecliptic plane). Shown are the unit vectors of the ecliptic co-ordinate system (subscript e) and the planetary co-ordinate system (p). Points P and Q are on the planet's orbit and are the same positions shown in fig. 1. The orientation of the orbit is described by the three angles Ω, i and ω and the orbital plane intersects the ecliptic along the dashed line.

Converting to ecliptic co-ordinates consists of three stages, each requiring a rotation. For the first rotation, we wish to choose new x and y axes, which are still in the orbital plane. Instead of choosing the perihelion as the direction of the *x*-axis (as in planetary co-ordinates), we choose the direction of the ascending node (fig. 3). The dashed line in fig. 3 is the same as that in fig. 2.

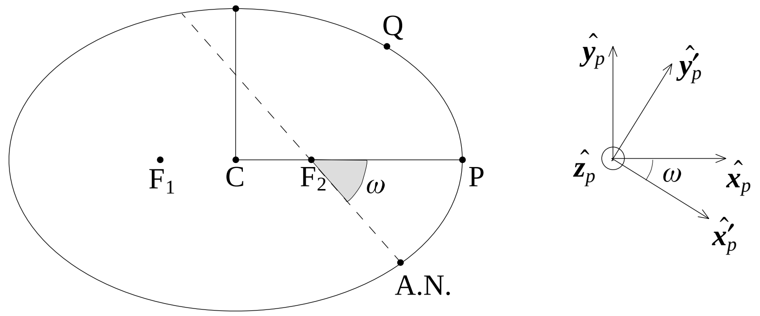


Fig. 3: The modified planetary co-ordinates. The plane of the paper is the plane of the orbit of the planet. The modified *x*-axis points from the Sun in the direction of the ascending node (primed unit vectors) instead of the direction of perihelion (unprimed vectors). Note: P is the perihelion and A.N. is the ascending node.

The angle between the old and new axes is ω, the argument of the perihelion. The conversion to the modified planetary co-ordinates (a vector expressed in this co-ordinate system is denoted by primed parentheses) requires simply an increment of the true anomaly. From eq. (6),

$$\boldsymbol{r}_p' = \begin{pmatrix} r\cos(w+\omega) \\ r\sin(w+\omega) \\ 0 \end{pmatrix}' . \tag{8}$$

The second intermediate co-ordinate system is a formed by a rotation (of angle i) of the modified planetary axes about $\hat{\boldsymbol{x}}_p'$. Let the new unit vectors be denoted by double primes (fig. 4). The vector $\boldsymbol{a}$ in fig. 4 has length ρ and points within the $\hat{\boldsymbol{y}}_p'$-$\hat{\boldsymbol{z}}_p'$ plane and in a direction an angle θ from $\hat{\boldsymbol{y}}_p'$. Although $\boldsymbol{a}$ itself is not important, it is useful is deriving the transformation. In the modified planetary co-ordinate system, $\boldsymbol{a}$ is,

$$\boldsymbol{a} = \begin{pmatrix} 0 \\ \rho\cos\theta \\ \rho\sin\theta \end{pmatrix}' , \tag{9}$$

and is used to find the double primed axes $\hat{\boldsymbol{y}}''$ and $\hat{\boldsymbol{z}}''$ by differentiating with respect to ρ and θ (respectively), choosing $\theta = -i$ and normalizing,

$$\hat{\boldsymbol{x}}'' = \hat{\boldsymbol{x}}_p' = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}' , \qquad \hat{\boldsymbol{y}}'' = \left.\frac{\partial \hat{\boldsymbol{a}}}{\partial \rho}\right|_{\theta=-i} = \begin{pmatrix} 0 \\ \cos i \\ -\sin i \end{pmatrix}' , \qquad \text{and } \hat{\boldsymbol{z}}'' = \left.\frac{\partial \hat{\boldsymbol{a}}}{\partial \theta}\right|_{\theta=-i} = \begin{pmatrix} 0 \\ \sin i \\ \cos i \end{pmatrix}' .$$

(10*a*) (10*b*) (10*c*)

The position in the double primed co-ordinate system is thus expressed as the position in the modified planetary co-ordinate system projected onto each of these unit vectors. This is most succinctly expressed as the matrix transformation,

$$\boldsymbol{r}'' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos i & -\sin i \\ 0 & \sin i & \cos i \end{pmatrix} \boldsymbol{r}_p'. \tag{11}$$

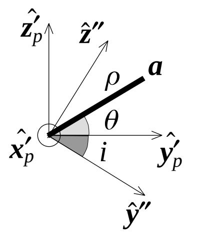


Fig. 4: The double-primed co-ordinate system expressed in the modified planetary co-ordinate system. ***a*** is a vector used only for the transformation.

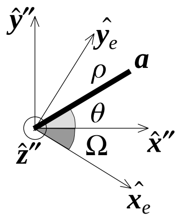


Fig. 5: The ecliptic co-ordinate system expressed in the double-primed co-ordinate system. The vector ***a*** serves an analogous purpose to that in fig. 4.

The final part of the transformation takes us to the ecliptic co-ordinate system, and requires a rotation of the co-ordinate axes of an angle Ω about $\hat{\boldsymbol{z}}''$ (fig. 5). In the double-primed co-ordinate system, the vector ***a*** is,

$$\boldsymbol{a} = \begin{pmatrix} \rho\cos\theta \\ \rho\sin\theta \\ 0 \end{pmatrix}''. \tag{12}$$

As before, the ecliptic unit vectors in the double-primed system are found by differentiation,

$$\hat{\boldsymbol{x}}_e = \left.\frac{\partial \hat{\boldsymbol{a}}}{\partial \rho}\right|_{\theta=\Omega} = \begin{pmatrix} \cos\Omega \\ -\sin\Omega \\ 0 \end{pmatrix}'', \qquad \hat{\boldsymbol{y}}_e = \left.\frac{\partial \hat{\boldsymbol{a}}}{\partial \theta}\right|_{\theta=\Omega} = \begin{pmatrix} \sin\Omega \\ \cos\Omega \\ 0 \end{pmatrix}'', \qquad \text{and} \quad \hat{\boldsymbol{z}}_e = \hat{\boldsymbol{z}}'' = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}''.$$

(13*a*) (13*b*) (13*c*)

Again, the transformation can be written as a matrix,

$$\boldsymbol{r}_e = \begin{pmatrix} \cos\Omega & -\sin\Omega & 0 \\ \sin\Omega & \cos\Omega & 0 \\ 0 & 0 & 1 \end{pmatrix} \boldsymbol{r}'' \tag{14}$$

The effective transformation from modified planetary co-ordinates to ecliptic co-ordinates is then (combining eqs. (11) and (14) where $\boldsymbol{r}_p'$ is the vector given as eq. (8)),

$$\boldsymbol{r}_e = \begin{pmatrix} \cos\Omega & -\sin\Omega\cos i & \sin\Omega\sin i \\ \sin\Omega & \cos\Omega\cos i & -\cos\Omega\sin i \\ 0 & \sin i & \cos i \end{pmatrix} \boldsymbol{r}_p' \tag{15}$$

3. The transformation to celestial co-ordinates

While the aim is to move to geocentric co-ordinates, all systems of co-ordinates used so far remain centred on the Sun (heliocentric). Once the planetary position, $\boldsymbol{r}_e$, has been computed for the chosen time, all that is required is a simple shift of origin to the Earth's position. This obviously requires us to know the position of the Earth at the same moment in time[2]. This is the subject of this section. Knowledge of the position of the Earth relative to the Sun is useful for other reasons too, enabling the position of the Sun in the sky, in addition to the planets, to be determined.

The vector marking the position of the Earth is calculated in the same way as for any of the planets. Since the Earth orbits in the ecliptic plane, few complicated axis rotations are required. For the Earth, $i = 0$, in which case the angles ω and Ω are ill defined. For a given orbit, they are measured from the point where the orbit crosses the ecliptic plane (fig. 2) - this is 'everywhere' if the orbit is always within the ecliptic. Their sum, $\varpi = \omega + \Omega$ (called the *longitude of the perihelion*), can be defined in this case, and indicates the angle between the vernal equinox and perihelion. To see this formally, apply the transformation matrix, eq. (15) with $i = 0$, to the vector, eq. (8). After application of some simple trigonometric identities, the position vector of the Earth in ecliptic co-ordinates is (E= Earth),

$$\boldsymbol{r}_e^E = \begin{Bmatrix} r\cos(w+\varpi) \\ r\sin(w+\varpi) \\ 0 \end{Bmatrix}, \tag{16}$$

where the curly parentheses indicate that the ecliptic co-ordinate system is used. The position vector of the planet relative to the Earth is the difference,

$$\boldsymbol{r}_g = \boldsymbol{r}_e - \boldsymbol{r}_e^E. \tag{17}$$

The celestial co-ordinates which we wish to adopt are the usual right ascension (RA) and declination (Dec.) parameters. These are akin to longitude and latitude familiar from our globe (both have the same equator).

The plane of the celestial equator is not coincident with that of the ecliptic. This is merely a statement that the Earth's axis of rotation (which defines the orientation of the RA/Dec. system of co-ordinates) is not normal to the ecliptic plane. Instead it is orientated at an angle ε, the *obliquity of the ecliptic* (fig. 6), making a further rotation of the axes is necessary. Proceeding in a similar way to the rotations made in section 2, the three new equatorial unit vectors (subscript *eq*) are written in the Earth-centred ecliptic system as,

2 Strictly, the position of the planet is needed at a slightly earlier time, owing to the finite speed of light. The further away the planet, the longer the delay. Since the maximum delay would be ~ 5 hours, and the distance travelled by the Earth and the planet in this time would be immeasurable at accuracies assumed for this work, we ignore this effect.

$$\hat{\boldsymbol{x}}_{eq} \;=\; \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}_g , \qquad \hat{\boldsymbol{y}}_{eq} \;=\; \frac{\partial \hat{\boldsymbol{a}}}{\partial \rho} \;=\; \begin{Bmatrix} 0 \\ \cos\varepsilon \\ -\sin\varepsilon \end{Bmatrix}_g \quad \text{and} \quad \hat{\boldsymbol{z}}_{eq} \;=\; -\frac{\partial \hat{\boldsymbol{a}}}{\partial \varepsilon} \;=\; \begin{Bmatrix} 0 \\ \sin\varepsilon \\ \cos\varepsilon \end{Bmatrix}_g ,$$

(18*a*) (18*b*) (18*c*)

with $\boldsymbol{a} = \{0, \rho\cos\varepsilon, \; -\rho\sin\varepsilon\}_g$ and the $\{\}_g$ notation implies that vectors are expressed in the ecliptic co-ordinate system centred on the Earth. This change of co-ordinates, which transforms the vector $\boldsymbol{r}_g$ (geocentric ecliptic co-ordinates) to $\boldsymbol{r}'_g$ (geocentric equatorial co-ordinates) is summarized by the matrix,

$$\boldsymbol{r}'_g \;=\; \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varepsilon & -\sin\varepsilon \\ 0 & \sin\varepsilon & \cos\varepsilon \end{pmatrix} \boldsymbol{r}_g. \tag{19}$$

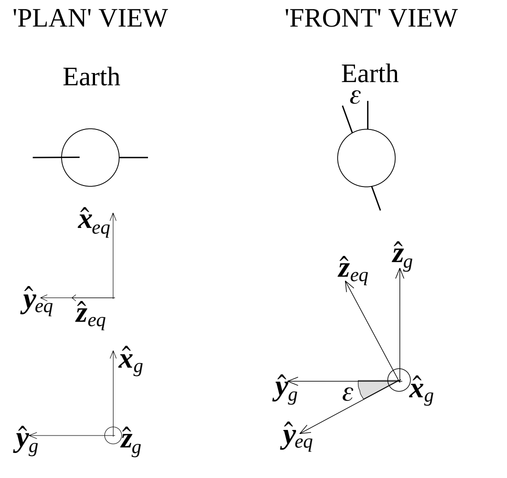


Fig. 6: The orientation of the Earth's axis with respect to the ecliptic plane. Two perspectives are shown. The view looking down on the Earth along the line which intersects the ecliptic at right angles is the 'plan view'. Looking along the line which joins the vernal equinox (in the $\hat{\boldsymbol{x}}_g$ direction) and the Earth is the 'front view'. Shown are the geocentric (subscript *g*) and the equatorial (subscript *eq*) co-ordinate systems. The latter is formed by a rotation of the former by an angle ε about $\hat{\boldsymbol{x}}_g$, and the equatorial axis, $\hat{\boldsymbol{z}}_{eq}$, points north along the Earth's axis.

The equatorial parameters RA and Dec. are derived from eq. (19) by simple trigonometry (the Earth-planet distance is also given),

$$\text{RA} \;=\; \begin{cases} \tan^{-1} \dfrac{\boldsymbol{r}'_g \cdot \hat{\boldsymbol{y}}_{eq}}{\boldsymbol{r}'_g \cdot \hat{\boldsymbol{x}}_{eq}} & \boldsymbol{r}'_g \cdot \hat{\boldsymbol{x}}_{eq} \;>\; 0 \\[2ex] \pi \;+\; \tan^{-1} \dfrac{\boldsymbol{r}'_g \cdot \hat{\boldsymbol{y}}_{eq}}{\boldsymbol{r}'_g \cdot \hat{\boldsymbol{x}}_{eq}} & \boldsymbol{r}'_g \cdot \hat{\boldsymbol{x}}_{eq} \;<\; 0, \end{cases} \tag{20}$$

$$\text{Dec.} \;=\; \tan^{-1} \frac{\boldsymbol{r}'_g \cdot \hat{\boldsymbol{z}}_{eq}}{\sqrt{(\boldsymbol{r}'_g \cdot \hat{\boldsymbol{x}}_{eq})^2 \;+\; (\boldsymbol{r}'_g \cdot \hat{\boldsymbol{y}}_{eq})^2}}, \tag{21}$$

$$\text{and} \qquad \text{distance} \;=\; \sqrt{(\boldsymbol{r}'_g \cdot \hat{\boldsymbol{x}}_{eq})^2 \;+\; (\boldsymbol{r}'_g \cdot \hat{\boldsymbol{y}}_{eq})^2 \;+\; (\boldsymbol{r}'_g \cdot \hat{\boldsymbol{z}}_{eq})^2} \,. \tag{22}$$

It is usual to convert RA into hours, minutes and seconds, and Dec. into degrees, minutes and seconds (the above are currently in radians). Note that for the derivation of eq. (20), it is essential to know the conventions related to the 'origin' and 'sense' of RA. At the vernal equinox (the Sun is directly 'above' the Earth in the plan view of fig. 6), the Sun has RA = 0. The RA becomes positive immediately thereafter (the Earth orbits the Sun in an anti-clockwise sense in the fig.).

4. The transformation to local horizontal co-ordinates

The celestial co-ordinate system is the standard framework in which most 'quasi-stationary' astronomical objects are catalogued. The celestial position of a planet (eqs. (20) and (21)) can be compared directly to the positions of stars in the vicinity. With the aid of a star chart, it can be located in the sky for a particular night. For an observer with a good degree of familiarity with the night sky, this 'method' of location is easy and practical for planets which are distinguished with the unaided eye. Otherwise, a more systematic technique is needed. In this section we present a more general means of location through a further transformation from celestial to local 'altitude-azimuth' co-ordinates (see below).

Each observer on Earth sees the sky from a different perspective depending upon time and on their location. The plane of the observer's horizon is tangential to the surface of the sphere of the Earth (fig. 7a), and rotates with the Earth once every sidereal day (1436.06817 minutes - about four minutes less than 24 hours). The transformation described below, which takes the celestial co-ordinates as input and yields the local alitude-azimuth angles specific to a given time and place can be applied equally as well to planets as stars, galaxies and nebulae. For a specific time and place on Earth in mind, the altitude-azimuth (or alt-azi) angles specify the position of an object relative to the horizon. Conventionally, the altitude, h, is the angle between the object and the horizon, and the azimuth, A, is the horizontal angle measured from north (fig. 7b).

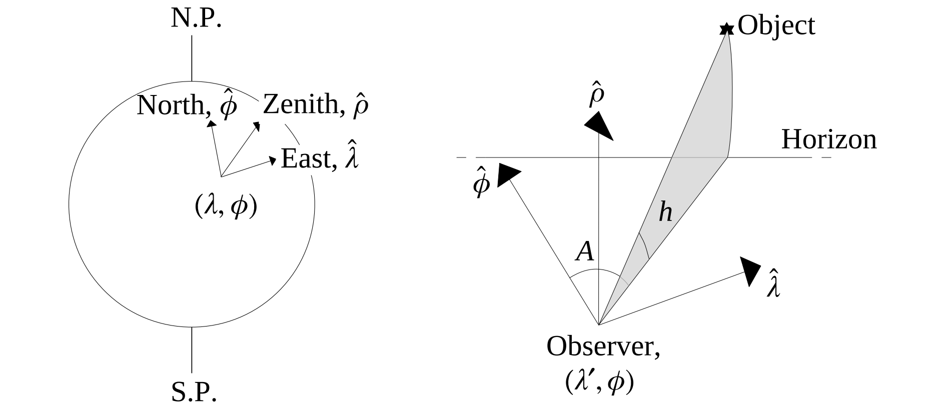


Fig. 7: (a) For an observer at a particular longitude and latitude, and at a specified time, three unit vectors $\hat{\phi}$, $\hat{\lambda}$, and $\hat{\rho}$ (corresponding to northerly, easterly and zenith directions respectively) can be defined (see text for their derivation). (b) By projecting the position vector of a star of planet onto these co-ordinates, trigonometry allows the altitude (h) and azimuth (A) angles to be computed for the local horizon of the observer.

In the context of the geocentric equatorial co-ordinate system defined in fig. 6, the position of the distant object with specified R.A. and Dec. parameters is,

$$\boldsymbol{r}'_g = \begin{pmatrix} \rho\cos\mathrm{Dec.}\cos\mathrm{R.A.} \\ \rho\cos\mathrm{Dec.}\sin\mathrm{R.A.} \\ \rho\sin\mathrm{Dec.} \end{pmatrix}. \quad (23)$$

Note that although an arbitrary distance parameter, ρ, has been used, our result (h and A) will be independent of ρ. In practice then it can be set to unity. (N.B. if the object in question is a planet computed using the formulae in section 3, its three position components of eq. (23) can be taken directly from eq. (19) rather than first converting to, and then back from R.A. and Dec. parameters.)

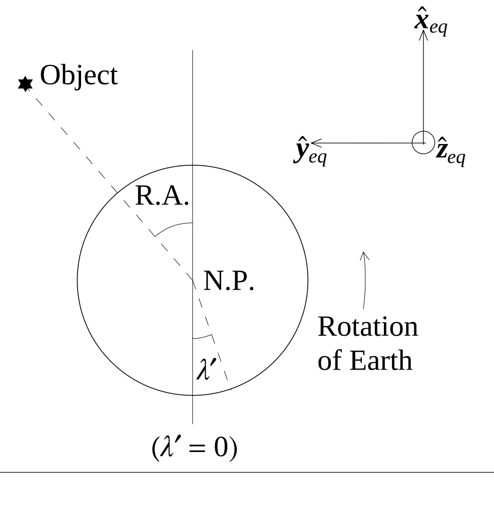
In order to make the transformation into the relevant local co-ordinates, the three unit vectors of fig. 7a need to be specified in the same co-ordinate system as the planet (geocentric equatorial co-ordinates). Let the location of the observer be at longitude, λ, and latitude, ϕ (each expressed in radians). We define the *effective* longitude of the observer, λ', relevant for time t,

$$\lambda' = \lambda + \frac{2\pi\,(t - t_{ref})}{\Delta t_s} \pmod{2\pi}. \quad (24)$$

Conceptually, λ' may be regarded as the longitude of an observer on a non-rotating Earth. Since the real Earth is rotating, the real observer is effectively in motion with respect to the non-rotating Earth. The parameters t_{ref} and Δt_s in eq. (24) are respectively, a reference time where an observer at $\lambda = 0$ would see an object of R.A. = 0 appear due north, and the length of the sidereal day. There are an infinite number of reference times to choose from, but possibly the simplest is to take the time of midnight on the day of the vernal equinox. The unit vectors will be derived from the position vector of the observer, $\boldsymbol{r}'_{obs}$ (fig. 8),

$$\boldsymbol{r}'_{obs} = \begin{pmatrix} -R\cos\phi\cos\lambda' \\ -R\cos\phi\sin\lambda' \\ R\sin\phi \end{pmatrix}, \quad (25)$$

Fig. 8: The relation between the R.A. = 0 half-plane and that of $\lambda' = 0$. The geocentric equatorial co-ordinates are also shown.



(R is the radius of the Earth). The northerly, easterly and zenith direction vectors can be be derived easily by partial differentiation followed by normalization,

$$\hat{\phi} = \frac{\partial \hat{\boldsymbol{r}}'_{obs}}{\partial \phi} = \begin{pmatrix} \sin\phi\cos\lambda' \\ \sin\phi\sin\lambda' \\ \cos\phi \end{pmatrix}, \quad \hat{\lambda} = \frac{\partial \hat{\boldsymbol{r}}'_{obs}}{\partial \lambda} = \begin{pmatrix} \sin\lambda' \\ -\cos\lambda' \\ 0 \end{pmatrix}, \quad \hat{\rho} = \frac{\partial \boldsymbol{r}'_{obs}}{\partial R} = \begin{pmatrix} -\cos\phi\cos\lambda' \\ -\cos\phi\sin\lambda' \\ \sin\phi \end{pmatrix}.$$

(26*a*) (26*b*) (26*c*)

The transformation which gives the 'directional position' of the object in the local co-ordinates, $\boldsymbol{r}_{local}$ is represented by the matrix consisting of eqs. (26) as its respective rows,

$$\boldsymbol{r}_{local} = \begin{pmatrix} \sin\phi\cos\lambda' & \sin\phi\sin\lambda' & \cos\phi \\ \sin\lambda' & -\cos\lambda' & 0 \\ -\cos\phi\cos\lambda' & -\cos\phi\sin\lambda' & \sin\phi \end{pmatrix} \boldsymbol{r}'_g. \tag{27}$$

The altitude and azimuth angles can be derived from this projection by simple trigonometry,

$$h = \tan^{-1} \frac{\boldsymbol{r}'_g \cdot \hat{\rho}}{\sqrt{(\boldsymbol{r}'_g \cdot \hat{\lambda})^2 + (\boldsymbol{r}'_g \cdot \hat{\phi})^2}} \tag{28}$$

$$\begin{aligned} A &= \tan^{-1} \frac{\boldsymbol{r}'_g \cdot \hat{\lambda}}{\boldsymbol{r}'_g \cdot \hat{\phi}} && \boldsymbol{r}'_g \cdot \hat{\phi} > 0 \\ &= \pi + \tan^{-1} \frac{\boldsymbol{r}'_g \cdot \hat{\lambda}}{\boldsymbol{r}'_g \cdot \hat{\phi}} && \boldsymbol{r}'_g \cdot \hat{\phi} < 0 \\ &= \pi/2 && \boldsymbol{r}'_g \cdot \hat{\phi} = 0 \text{ and } \boldsymbol{r}'_g \cdot \hat{\lambda} > 0 \\ &= -\pi/2 && \boldsymbol{r}'_g \cdot \hat{\phi} = 0 \text{ and } \boldsymbol{r}'_g \cdot \hat{\lambda} < 0 \end{aligned} \tag{29}$$

5. Determining orbital parameters from observations

As discussed in the earlier sections, the future positions of a planet can be predicted if its six orbital parameters (commonly a, e, i, Ω, ϖ and ε) are known. Given also those parameters of the observer (the Earth), then we can infer the location of the planet in the sky at any future time for which the two-body approximation holds. In this section we will discuss how the inverse problem is solved, that is, the determination of a planet's orbital parameters from a set of suitable observations.

In this section we will introduce a simple notation that is convenient to the inverse problem. The six orbital parameters (as above) are assembled into a six-element vector $\boldsymbol{x}$ (known also as the *state*). It is this vector that we wish to determine. This will be done by first guessing this state and then refining the guess iteratively, by fitting to the observations, using Gauss' method of least squares. Let there be N observations consisting of alt/azi or RA/dec pairs, each pair pertaining to a given time (we require a minimum of $N = 3$ for the problem to be well-posed). The observations are contained in the vector $\boldsymbol{y}$ of $2N$ elements. All observations have uncertainties. Let these be specified as variances and occupy corresponding diagonal elements of the diagonal matrix $\mathbf{R}$.

The actual observations are distinguished from the so-called 'model observations'. These are represented as a similar vector to $\boldsymbol{y}$, but are those 'observations' that are predicted from the orbital parameters in $\boldsymbol{x}$ by the prescription in the preceding sections. The idea is to vary $\boldsymbol{x}$ until the sum of weighted-mean-square difference between the observations and the model observations is minimized.

For clarity, all of the details of the previous sections will be labelled by the non-linear vector operator, $\boldsymbol{H}[\boldsymbol{x}]$. $\boldsymbol{H}[\boldsymbol{x}]$ is the model observation operator ($2N$ elements - as $\boldsymbol{y}$) which takes as its primary argument the set of orbital parameters (other inputs, including the time that each actual observation is made and the type of observation (alt/azi or RA/dec), are implicit). $\boldsymbol{H}[\boldsymbol{x}]$ is also known as the *forward model*, which we know how to solve. The inverse problem actually involves multiple use of the forward model and is posed in the context of minimizing the *cost function*, J,

$$J = \frac{1}{2}(\boldsymbol{y} - \boldsymbol{H}[\boldsymbol{x}])^T \mathbf{R}^{-1} (\boldsymbol{y} - \boldsymbol{H}[\boldsymbol{x}]). \tag{30}$$

The remainder of this section will deal with how $\boldsymbol{x}$ can be varied in such a way as to minimize J.

Since $\boldsymbol{H}[\boldsymbol{x}]$ is non-linear, we need to linearize it,

$$\boldsymbol{H}[\boldsymbol{x}] = \boldsymbol{H}[\boldsymbol{x}_0] + \mathbf{H}(\boldsymbol{x} - \boldsymbol{x}_0). \tag{31}$$

$\mathbf{H}$ is the linearization of $\boldsymbol{H}$ about a guess state $\boldsymbol{x}_0$. Note that $\mathbf{H}$ (in roman rather than italic typeface) is a $2N \times 6$ matrix which we denote by $\mathrm{d}\boldsymbol{H}/\mathrm{d}\boldsymbol{x}$. In matrix notation (^{T} denotes the adjoint or transpose) this is,

$$\mathbf{H} = \frac{\mathrm{d}\boldsymbol{H}}{\mathrm{d}\boldsymbol{x}} = \left(\left(\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{x}}\right)^T \boldsymbol{H}^T\right)^T = \left(\begin{pmatrix} \partial/\partial x_1 \\ \partial/\partial x_2 \\ \ldots \\ \partial/\partial x_6 \end{pmatrix} (H_1[\boldsymbol{x}] \;\; H_2[\boldsymbol{x}] \;\; \ldots \;\; H_{2N}[\boldsymbol{x}])\right)^T. \tag{32}$$

We linearize the whole cost function about $\boldsymbol{x}_0$ by substituting Eq. (31) into (30),

$$J = \frac{1}{2}(\boldsymbol{y} - \boldsymbol{H}[\boldsymbol{x}_0] - \mathbf{H}(\boldsymbol{x} - \boldsymbol{x}_0))^T \mathbf{R}^{-1} (\boldsymbol{y} - \boldsymbol{H}[\boldsymbol{x}_0] - \mathbf{H}(\boldsymbol{x} - \boldsymbol{x}_0)). \tag{33}$$

J is now purely quadratic in $\boldsymbol{x}$. To simplify the appearance of this expression, we identify two residuals. Let $\boldsymbol{y} - \boldsymbol{H}[\boldsymbol{x}_0] = \delta\boldsymbol{y}$ and $\boldsymbol{x} - \boldsymbol{x}_0 = \delta\boldsymbol{x}$. Then,

$$J = \frac{1}{2}(\delta\boldsymbol{y} - \mathbf{H}\delta\boldsymbol{x})^T \mathbf{R}^{-1}(\delta\boldsymbol{y} - \mathbf{H}\delta\boldsymbol{x}). \tag{34}$$

The gradient of J with respect to $\delta\boldsymbol{x}$ is a vector,

$$\nabla_{\delta x} J = \left(\frac{\mathrm{d}J}{\mathrm{d}\delta\boldsymbol{x}}\right)^T = -\mathbf{H}^T\mathbf{R}^{-1}(\delta\boldsymbol{y} - \mathbf{H}\delta\boldsymbol{x}), \tag{35}$$

and the second derivative (known as the Hessian) is,

$$\text{Hessian} = \left(\frac{\mathrm{d}}{\mathrm{d}\delta\boldsymbol{x}}\right)^T \frac{\mathrm{d}J}{\mathrm{d}\delta\boldsymbol{x}} = \mathbf{H}^T\mathbf{R}^{-1}\mathbf{H}. \tag{36}$$

Preconditioning the problem

Variational problems are quite difficult to solve and often some mathematical preparation of the system has to be made. This is called *preconditioning*.

It is desirable for the Hessian to resemble the identity matrix, **I** (the reason for this is explained in the next subsection). Of course, in general the Hessian will not be the identity matrix, nor will it be either diagonal or have unit determinant. We would like to make a transformation to a new vector space in which the Hessian is the identity matrix. This is a stage of the preconditioning process. Although this complicates the problem at this stage, it simplifies the minimization later on. Let $\mathbf{U}^{-1}$ be the transformation which moves from $\delta\boldsymbol{x}$ to a new $\delta\boldsymbol{u}$ space (we wish to do the minimization in $\delta\boldsymbol{u}$ space). $\mathbf{U}^{-1}$ is represented as a matrix (as is its inverse, **U**, which we are more concerned with),

$$\delta\boldsymbol{u} = \mathbf{U}^{-1}\delta\boldsymbol{x} \tag{37}$$

$$\delta\boldsymbol{x} = \mathbf{U}\delta\boldsymbol{u} \tag{38}$$

$$= \mathbf{U}_2\mathbf{U}_1\delta\boldsymbol{u}. \tag{39}$$

In the last line the transformation has been split into parts 1 and 2, $\mathbf{U} = \mathbf{U}_2\mathbf{U}_1$, which shall be exploited soon. Inserting **U** into the cost function (Eq. (34)) gives,

$$J = \frac{1}{2}(\delta\boldsymbol{y} - \mathbf{H}\mathbf{U}\delta\boldsymbol{u})^T \mathbf{R}^{-1}(\delta\boldsymbol{y} - \mathbf{H}\mathbf{U}\delta\boldsymbol{u}), \tag{40}$$

in $\delta\boldsymbol{u}$-space. Expressions for the first and second derivatives of the cost function with respect to $\delta\boldsymbol{u}$ can be derived from those with respect to $\delta\boldsymbol{x}$ (Eqs. (35) and (36)) using the transformation (Eq. (38)) and the generalised chain rule result [6],

$$\left(\frac{\mathrm{d}J}{\mathrm{d}\delta\boldsymbol{u}}\right)^T = \mathbf{U}^T\left(\frac{\mathrm{d}J}{\mathrm{d}\delta\boldsymbol{x}}\right)^T \tag{41}$$

The gradient and Hessian in $\delta\boldsymbol{u}$ space are then,

$$\nabla_{\delta u} J = \left(\frac{\mathrm{d}J}{\mathrm{d}\delta\boldsymbol{u}}\right)^T = -\mathbf{U}^T\mathbf{H}^T\mathbf{R}^{-1}(\delta\boldsymbol{y} - \mathbf{HU}\delta\boldsymbol{u}) \tag{42}$$

$$\text{Hessian} = \left(\frac{\mathrm{d}}{\mathrm{d}\delta\boldsymbol{u}}\right)^T \frac{\mathrm{d}J}{\mathrm{d}\delta\boldsymbol{u}} = \mathbf{U}^T\mathbf{H}^T\mathbf{R}^{-1}\mathbf{HU}. \tag{43}$$

We would like to find the transformation **U** that will, in $\delta\boldsymbol{u}$ space, yield a Hessian that is the identity matrix. A transformed Hessian with such a property is well conditioned and is thus simple to deal with when we come to minimize the cost function. In order to find **U**, we exploit what is already known about transformations that diagonalize a symmetric matrix. Given the Hessian in $\delta\boldsymbol{x}$ space, we know that it can be diagonalized with the special matrix **Y**,

$$\mathbf{Y}\left(\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H}\right)\mathbf{Y}^T = \Lambda. \tag{44}$$

Here, the matrix within brackets is the Hessian in the $\delta\boldsymbol{x}$ representation and the rows of matrix **Y** are formed by the eigenvectors of the Hessian (in $\delta\boldsymbol{x}$ space), and Λ is a diagonal representation of the Hessian (the diagonal elements are the eigenvalues). We can complete the transformation (to make the transformed Hessian unitary) by pre and post multiplying each side of Eq. (44) by $\Lambda^{-1/2}$,

$$\begin{array}{ccc} \Lambda^{-1/2}\mathbf{Y}\left(\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H}\right)\mathbf{Y}^T\Lambda^{-1/2} & = & \Lambda^{-1/2}\Lambda\Lambda^{-1/2} \\ \downarrow & = & \downarrow \\ \Lambda^{-1/2}\mathbf{Y}\left(\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H}\right)\left(\Lambda^{-1/2}\mathbf{Y}\right)^T & = & I, \end{array} \tag{45}$$

($\Lambda^{-1/2} = \left(\Lambda^{-1/2}\right)^T$). We wish to find out how transformations **Y** and $\Lambda^{-1/2}$ - which we can find by diagonalizing the Hessian - are related to **U**. Equate the above to the form of the Hessian in $\delta\boldsymbol{u}$ space, Eq. (43),

$$\mathbf{U}^T\left(\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H}\right)\mathbf{U} = \Lambda^{-1/2}\mathbf{Y}\left(\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H}\right)\left(\Lambda^{-1/2}\mathbf{Y}\right)^T, \tag{46}$$

and it becomes obvious (by comparison of each side) that we may choose,

$$\begin{array}{rcl} \mathbf{U} & = & \left(\Lambda^{-1/2}\mathbf{Y}\right)^T \\ \downarrow & = & \downarrow \\ \mathbf{U}_2\mathbf{U}_1 & = & \mathbf{Y}^T\Lambda^{-1/2} \end{array}$$

$$\text{i.e.}\quad \mathbf{U}_2 = \mathbf{Y}^T \tag{47}$$

$$\text{and}\quad \mathbf{U}_1 = \Lambda^{-1/2}. \tag{48}$$

Figure 9 shows cross sections through the non-linear function J (Eq. (30)) centred on the orbital parameters for Jupiter using 11 synthetic observations spanning 40 years. The respective variation with respect to each parameter is featured in panels "a" - "f". The structure of J (particularly as $\boldsymbol{x}_1 = a$ is varied) is complicated. This presents a number of difficulties. The state that we wish to converge towards (the global minimum) is labelled "A" in each panel. There are however other minima (e.g. "B" in Fig. 9a) that the procedure may find, which are erroneous.

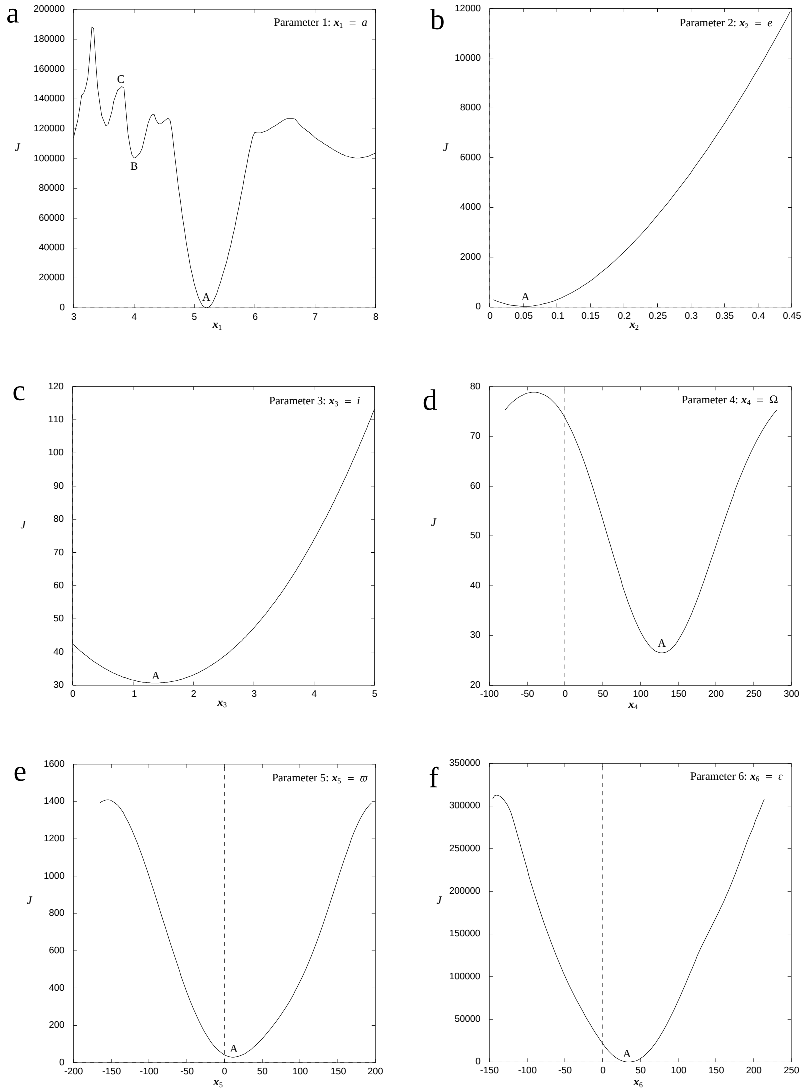


Fig. 9: Structure of the cost function, J. Each parameter, $\boldsymbol{x}_1$ to $\boldsymbol{x}_6$, is varied separately in panels "a" to "f" respectively. The parameters centre on those relevant to Jupiter (see section 7), and 11 synthetic observations (of the alt/azi variety) have been used spanning a 40-year period. The global minimum is shown as "A" in each panel. The state $\boldsymbol{x}$ at this point inverts the problem. "B" and "C" are erroneous stationary points discussed in the text.

Furthermore, the 'minimization' procedure outlined in the next subsection actually searches for stationary points and left freely will not necessarily go 'downhill'. Convergence may occur towards a local maximum (e.g. "C" in Fig. 9a).

Whether the wrong minimum or a maximum is approached depends upon the initial guess, $\boldsymbol{x}_0$. The linearization process fits a six-dimensional paraboloid to the local shape of the true cost function. If a maximum is 'captured' by the linearization then one or more of the diagonal elements of the Hessian, Eq. (36), will be negative. Consequently, one or more of the eigenvalues (in Λ) will be negative and part of the **U**-transformation ($\mathbf{U}_1$ in Eq. (48)) will become imaginary (we require that the square-root of the eigenvalue matrix is real).

These problems are overcome by ensuring (prior to finding the **U**-transformation) that $\boldsymbol{x}_0$ will capture the global minimum. The guess, $\boldsymbol{x}_0$ is adjusted by first performing two one-dimensional searches for the smallest value of J with respect only to parameter "i" and then only to parameter "a". All other parameters are angular (and hence periodic) and if the Hessian contains a negative diagonal element then the parameter is shifted by π radians. This is actually the initial part of the preconditioning process.

<u>Minimization</u>

The minimization is done in $\delta\boldsymbol{u}$ space. Expanding J as a Taylor series about $\delta\boldsymbol{u} = 0$,

$$\begin{aligned} J(\delta\boldsymbol{u}) &= J(0) + \left.\frac{\mathrm{d}J}{\mathrm{d}\delta\boldsymbol{u}}\right|_0 \delta\boldsymbol{u} + \frac{1}{2}\delta\boldsymbol{u}^T \left.\left(\left(\frac{\mathrm{d}}{\mathrm{d}\delta\boldsymbol{u}}\right)^T \frac{\mathrm{d}J}{\mathrm{d}\delta\boldsymbol{u}}\right)\right|_0 \delta\boldsymbol{u} \\ &= J(0) + \left.\frac{\mathrm{d}J}{\mathrm{d}\delta\boldsymbol{u}}\right|_0 \delta\boldsymbol{u} + \frac{1}{2}\delta\boldsymbol{u}^T\delta\boldsymbol{u}, \end{aligned} \tag{49}$$

where the value $J(\delta\boldsymbol{u} = 0) = J(\delta\boldsymbol{x} = 0) = J(\boldsymbol{x}_0)$. The third (quadratic) term has been simplified in the last line as the Hessian is the identity matrix in $\delta\boldsymbol{u}$-space (it is for the purpose of this simplification that we introduced the **U**-transform in the previous subsection). Now differentiating Eq. (49) with respect to the deviation $\delta\boldsymbol{u}$, and setting to zero for the stationary point,

$$\left.\left(\frac{\mathrm{d}J}{\mathrm{d}\delta\boldsymbol{u}}\right)^T\right|_{\delta u} = \left.\left(\frac{\mathrm{d}J}{\mathrm{d}\delta\boldsymbol{u}}\right)^T\right|_0 + \delta\boldsymbol{u} = 0, \tag{50}$$

allows one to find the displacement $\delta\boldsymbol{u}$ which minimizes the cost function,

$$\delta\boldsymbol{u} = -\left.\left(\frac{\mathrm{d}J}{\mathrm{d}\delta\boldsymbol{u}}\right)^T\right|_0. \tag{51}$$

Recall the form of the gradient term Eq. (42), which makes this displacement,

$$\delta\boldsymbol{u} = \mathbf{U}^T\mathbf{H}^T\mathbf{R}^{-1}\delta\boldsymbol{y}. \tag{52}$$

This is known and closes the expression for $\delta\boldsymbol{u}$. This procedure, which searches for minima, would have been more difficult than it is if the

Hessian had not been preconditioned beforehand. The minimum (in the linearization) thus occurs at a deviation, $\delta\boldsymbol{u}$, from the linearization point. This can be transformed into $\delta\boldsymbol{x}$ space via Eq. (38). We then gain a better estimate for the set of orbital parameters,

$$\boldsymbol{x} = \boldsymbol{x}_0 + \mathbf{U}\delta\boldsymbol{u}. \tag{53}$$

Although the linearized inverse problem has been solved, the true cost function, Eq. (30) is not quadratic. It is assumed that the non-linear problem is inverted by replacing $\boldsymbol{x}_0$ with $\boldsymbol{x}$ from Eq. (53) and repeating the whole process in an iterative fashion. The solution is found when some convergence criterion is satisfied (when $\delta\boldsymbol{u} \rightarrow 0$ e.g.).

Uncertainties

An important by-product of this variational procedure is that the uncertainties of the analysed parameters can also be found. A further computation of $\mathbf{H}$ at the solution point allows a final calculation of the Hessian, $\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H}$. The diagonal elements of the Hessian are the variances of the solution. The smaller the variances, the more accurate the result, and can be improved by repeating the inverse problem with a greater number of observations.

Algorithm

The algorithm for solving this inverse problem is contained in the following flow chart.

Read-in observations, $\boldsymbol{y}$, the observation times and types (RA/dec or alt/azi), and the diagonal elements of $\mathbf{R}$.

↓

Set $n = 0$ (n it the iteration number) and guess the solution, $\boldsymbol{x}_1$.

↓

{*1} Increment n.

↓

{*2} Do a partial minimization over "i" parameter and update $\boldsymbol{x}_n$.

↓

Do a partial minimization over "a" parameter and update

x_n.

↓

Calculate the predicted observations, $\boldsymbol{H}[\boldsymbol{x}_n]$.

↓

Calculate the full cost function as a diagnostic.

↓

Define $\delta\boldsymbol{y} = \boldsymbol{y} - \boldsymbol{H}[\boldsymbol{x}_n]$.

↓

Linearize the forward model in physical space about $\boldsymbol{x}_n$.

↓

Calculate the Hessian in physical space at the linearization state ($\boldsymbol{x}_n$).

↓

Check for negative diagonal elements of the Hessian. The partial minimization with respect to "i" and "a" will guarantee that diagonal element corresponding to these will be positive. If a negative diagonal element is found for another element, adjust its value by π radians and return to {*2} above.

↓

Diagonalize the Hessian and determine the co-ordinate transforms, $\mathbf{U}_1$ and $\mathbf{U}_2$ to give a unitary Hessian.

$$\begin{aligned} \mathbf{U} &= \mathbf{U}_2\mathbf{U}_1 \\ \delta\boldsymbol{x} &= \mathbf{U}\delta\boldsymbol{u} \\ &= \mathbf{U}_2\mathbf{U}_1\delta\boldsymbol{u} \end{aligned}$$

↓

Calculate the gradient in $\delta\boldsymbol{u}$ space, $\nabla_{\delta u}J$ ($\delta\boldsymbol{u}$ is initially zero).

↓

Update $\delta\boldsymbol{u}$ that minimizes J in the linearization.

↓

Calculate the new state vector $\boldsymbol{x}_{n+1} = \boldsymbol{x}_n + \mathbf{U}\delta\boldsymbol{u}$.

↓

← Go back to {*1} if the solution has not sufficiently converged.

6. Solar System Data

Mass of the Sun, $M_s = 1.989 \times 10^{30}$ kg [4].

Obliquity of the ecliptic, $\varepsilon = 23° \, 26' \, 21''$ (2000 Jan 1) [3].

Table 1: Orbital constants [5]

Planet	a_0 (AU)	e_0	i_0 (°)	Ω_0 (°)	ϖ_0 (°)	ε_0 (°)
Mercury	0.38709893	0.20563069	7.00487	48.33167	77.45645	252.25084
Venus	0.72333199	0.00677323	3.39471	76.68069	131.53298	181.97973
Earth	1.00000011	0.01671022	0.00005	-11.26064	102.94719	100.46435
Mars	1.52366231	0.09341233	1.85061	49.57854	336.04084	355.45332
Jupiter	5.20336301	0.04839266	1.30530	100.55615	14.75385	34.40438
Saturn	9.53707032	0.05415060	2.48446	113.71504	92.43194	49.94432
Uranus	19.19126393	0.04716771	0.76986	74.22988	170.96424	313.23218
Neptune	30.06896348	0.00858587	1.76917	131.72169	44.97135	304.88003
Pluto	39.48168677	0.24880766	17.14175	110.30347	224.06676	238.92881

Table 2: Corrections (linear in time) [5]

Planet	$\dot{a}$ (AU/cy)	$\dot{e}$ (/cy)	$\dot{i}$ ("/cy)	$\dot{\Omega}$ ("/cy)	$\dot{\varpi}$ ("/cy)	$\dot{\varepsilon}$ ("/cy)
Mercury	0.00000066	0.00002527	-23.51	-446.30	573.57	538101628.29
Venus	0.00000092	-0.00004938	-2.86	-996.89	-108.80	210664136.06
Earth	-0.00000005	-0.00003804	-46.94	-18228.25	1198.28	129597740.63
Mars	-0.00007221	0.00011902	-25.47	-1020.19	1560.78	68905103.78
Jupiter	0.00060737	-0.00012880	-4.15	1217.17	839.93	10925078.35
Saturn	-0.00301530	-0.00036762	6.11	-1591.05	-1948.89	4401052.95
Uranus	0.00152025	-0.00019150	-2.09	-1681.40	1312.56	1542547.79
Neptune	-0.00125196	0.0000251	-3.64	-151.25	-844.43	786449.21
Pluto	-0.00076912	0.00006465	11.07	-37.33	-132.25	522747.90

Elements are referenced to mean ecliptic and equinox of J2000 at the J2000 epoch (2451545.0 JD).

The tables give the six orbital parameters:

a, length of the semi-major axis, e, eccentricity,

i, inclination, Ω, longitude of the ascending node,

ϖ, longitude of the perihelion, and ε, mean longitude,

(table 1) and linear corrections of each due to perturbations (table 2). To estimate the time dependent value of a parameter, χ, the initial value ,χ_0, and the rate of change, $\dot{\chi}$, taken from the table give,

$$\chi(t) = \chi_0 + \gamma\dot{\chi}\,\frac{t - t_0}{36525},$$

where $\chi_0 \equiv \chi(t_0)$ (t_0 is the epoch), t and t_0 are in days, and 36525 is the number of days in a century. The parameter γ is just a conversion factor, ie $\gamma = 1/3600$ for angular quantities, and $\gamma = 1$ otherwise. Note that in the case of the mean longitude, the result $\varepsilon_0 + \gamma\dot{\varepsilon}(t - t_0)/36525$ includes the effect of the mean motion, n in eq. (7). The mean anomaly, M, (in degrees) is then calculated from the table as (use this instead of eq. (7)),

$$M = \varepsilon_0 - \varpi_0 + (\dot{\varepsilon} - \dot{\varpi})\,\frac{t - t_0}{36525 \times 3600}.$$

7. Glossary (see also [3])

Aphelion The point of an elliptical orbit which is furthest away from the occupied focus.

Argument of perihelion The angle, measured in the orbit of the planet about the occupied focus, between the ascending node and the perihelion (ω in fig. 2).

Ascending node This is the point in the planetary orbit which crosses the ecliptic from below (fig. 2).

Celestial co-ordinates A system of co-ordinates which maps objects onto the surface of the celestial sphere as viewed from the Earth (at its centre) The usual celestial co-ordinates is of equatorial type. This is a system of spherical polar geocentric co-ordinates. The two co-ordinates are right ascension (RA, which is analogous to longitude) and declination (Dec.., analogous to latitude). The celestial equator is a projection of the Earth's equator on to the celestial sphere. The zero of RA is the direction of the vernal equinox.

Cost function A scalar measure of the misfit between a set of observations and a corresponding set that has been calculated according to a model (forward model). The cost function is a tool used generally in variational inverse modelling. The inverse model is solved when the model parameters (in the astronomical example, Eq. (30), the six orbital parameters) minimize the cost function.

Eccentric anomaly The angle (E in fig. 1) formed between perihelion and a given point on the elliptical orbit in question, measured about the centre of the ellipse.

Eccentricity The parameter which describes how far an ellipse has deviated from a circle (see eq. (1)). The eccentricity multiplied by the length of the semi-major axis yields the separation between the centre of the ellipse and one of the foci. A circle has the eccentricity of zero.

Ecliptic co-ordinates A set of co-ordinates for which the x-y plane lies in the plane of the Earth's orbit. The centre of the system is at the position of the Sun. The x-axis points in the direction of the vernal equinox, and the z-axis points perpendicular to the orbital plane (looking along z, the Earth's orbit is clockwise) and the y-axis is perpendicular to the other two axes in a right-handed sense. See fig. 2.

Ellipse One of the possible paths traced-out by gravitational orbital motion between two bodies. The other possible orbits are parabolas and hyperbolas, the ellipse being the bound (closed) orbit.

Equatorial co-ordinates See e.g. under 'celestial co-ordinates'.

Focus An ellipse has two foci (fig. 1). The orbit of a minor mass about a major body is an ellipse and the major mass would be positioned at one of the foci (called the occupied focus).

Forward model A forward model is a set of solvable mathematical rules that acts on one set of parameters to give another set. It is to be distinguished from the inverse model, which is often insolvable or difficult to solve. In any case, the solution to the inverse model can be estimated in a variational way by minimizing a cost function which is actually defined in terms of the forward model. The astronomical forward model developed in these notes takes the six orbital parameters of a planet and predicts its position at a specified time.

Geocentric co-ordinates A system of co-ordinates centred on the Earth.

Heliocentric co-ordinates A system of co-ordinates centred on the Sun.

Inclination The angle, i, between the planes (and normals to the planes) of a planet and the ecliptic (fig. 2).

Longitude of the ascending node The angle between the vernal equinox and the ascending node of a planets orbit centred on the Sun (Ω in fig. 2).

Longitude of the perihelion The sum of the argument of the perihelion, ω, and the longitude of the ascending node, Ω (symbol ϖ). ω and Ω each have little meaning for orbits which have $i = 0°$ (such as the Earth) or $i = 180°$, but in which cases their sum is meaningful. This is the angle between the vernal equinox and the perihelion.

Mean anomaly The angle, M (eqs. (2) and (3)), between perihelion and a point on a fictitious orbit, measured about the centre of the real elliptical orbit (of the planet). The fictitious orbit is the circular orbit which has the same orbital period as the planet, and the point is that of the fictitious body (travelling at uniform speed) after a given amount of time since perihelion.

Mean longitude The mean longitude, L is defined as the sum of the longitude of the perihelion and the mean anomaly, M. M is time dependent, but if the mean longitude is to be quoted as an alternative parameter to the longitude of the perihelion, then its value should be chosen at a specified epoch. In this case, the mean longitude adopts the symbol ε and should be quoted together with the chosen date of epoch.

Mean motion The angular frequency, *n* (eq. (3)), of a fictitious body in a circular orbit which has the same orbital period as the real planet.

Obliquity of the ecliptic This is the angle between the axis of the Earth and the normal to the ecliptic plane (symbol ε). It is the same angle between the planes of the celestial equator and the ecliptic.

Perihelion The point of an elliptical orbit which is closest to the occupied focus (fig. 1).

Planetary co-ordinates A convenient co-ordinate system used to describe the position of a planet in its plane. This system of co-ordinates is centred on the Sun. The x-axis points in the direction of the perihelion, and the z-axis points perpendicular to the orbital plane (looking along z, the planet's orbit is clockwise) and the y-axis is perpendicular to the other two axes in a right-handed sense.

Semi-major axis The line between the centre of an ellipse and the point of perihelion (fig. 1).

Semi-minor axis The line between the centre of an ellipse and the point touching the ellipse by moving in a direction perpendicular to the semi-major axis (fig. 1).

True anomaly The angle, *w* (eq. 4), formed between perihelion and a given point on the elliptical orbit in question, measured about the occupied focus.

Vernal Equinox (or *first point of Aries*) A point on the Earth's orbit which is defined according to the orientation of the tilt of the Earth's axis. The point is marked by the position of the Earth at that time when the Earth's axis lies in the tangent plane of the orbit, and the Sun (to an Earthbound observer) appears to move northwards. For the current epoch, the vernal equinox occurs on March 21st each year (fig. 2). The vernal equinox defines the zero of the celestial co-ordinate system.

8. References

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